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On the Jacobson radical of graded rings

A.V. Kelarev

Abstract. All commutative semigroups S are described such that the Jacobson radical is homogeneous in each ring graded by S.

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In the theory of rings, many structure results were obtained with the use of radicals; and the Jacobson radical seems to be the most efficient. The concept of a radical ϱ enables one to reduce various problems concerning an arbitrary ring R to the corresponding questions on the rings $\varrho(R)$ and $R/\varrho(R)$ which are radical and semisimple, respectively. For the applications of a well-known radical ϱ to the study of graded rings, it is essential to know when it is homogeneous, because in that case both $\varrho(R)$ and $R/\varrho(R)$ are graded as well. In [1] abelian groups G were described such that the Jacobson radical is homogeneous in every G-graded ring. The aim of the present paper is to describe those commutative semigroups S such that the Jacobson radical is S-homogeneous.

The radicals of semigroup-graded rings have been investigated by a number of authors for several classes of semigroups. A few results of a graded nature have already contributed to the solutions of some problems on semigroup rings. For instance, the theorems of [1] and [15] play important roles in the description of the Jacobson radical $J(R[S])$ for a commutative S, see [9]; the results of [3] and [4] were applied to the study of semigroup rings satisfying polynomial identities in [12]. The homogeneity of radicals in a semigroup-graded ring was considered in [1], [5], [7], [8], [10], [14].

Let S be a semigroup. An associative ring R is called an S-graded ring if there exist additive subgroups R_s of R indexed by the elements $s \in S$ such that $R = \bigoplus_{s\in S} R_s$ is a direct sum and $R_sR_t \subseteq R_{st}$ for all s, t. The Jacobson radical J is said to be S-homogeneous if $J(R) = \bigoplus_{s\in S}(J(R) \cap R_s)$ for each $R = \bigoplus_{s\in S} R_s$.

Theorem. *Let S be a commutative semigroup. The Jacobson radical is S-homogeneous if and only if S is embeddable in a torsion-free abelian group.*

Proof: The 'if' part is an immediate consequence of the results of [1]. Indeed, assume that S is contained in a torsion-free abelian group G. Take any ring $R = \bigoplus_{s\in S} R_s$. Setting $S_g = 0$ for $g \in G \setminus S$, we get $R = \bigoplus_{g\in G} R_g$. It was shown in [1] (see also [14]) that the Jacobson radical is G-homogeneous. Therefore $J(R) = \bigoplus_{g\in G}(J(R) \cap R_g) = \bigoplus_{s\in S}(J(R) \cap R_s)$. Thus J is S-homogeneous.

For the proof of necessity we need the following definitions. A commutative semigroup S is said to be separative if $s, t \in S$, $s^2 = st = t^2$ imply $s = t$. The least separative congruence ξ on S is the least congruence such that S/ξ is separative. Explicitly (cf. [2, § 4.3])

$$\xi = \{(s,t) \mid s^n t = s^{n+1},\ t^n s = t^{n+1} \text{ for a natural } n\}.$$

A semigroup S is p-separative for a prime p, if $s, t \in S$, $s^p = t^p$ imply $s = t$. The least p-separative congruence on S is denoted by ξ_p. It is known (cf. [11]) that

$$\xi_p = \{(s,t) \mid s^{p^n} = t^{p^n} \text{ for a natural } n\}.$$

If A is an ideal of R, η is a congruence on S, then the ideal of $R[S]$ consisting of all sums $\sum_{i=1}^{n} a_i(s_i - t_i)$, where $a_i \in A$, $(s_i, t_i) \in \eta$, is denoted by $I(A, S, \eta)$. A commutative semigroup B is called a semilattice, if it consists of idempotents. S is said to be a semilattice B of its semigroups S_b, $b \in B$, if $S = \bigcup_{b \in B} S_b$, $S_a \cap S_b = \emptyset$ whenever $a \neq b$, and $S_a \subseteq S_b$ for any $a, b \in B$. Let $\leq$ denote the natural partial order on B defined by the rule $a \leq b \Leftrightarrow ab = a$.

Now let us prove the 'only if' part. Assume that J is S-homogeneous. If F is a field of characteristic zero, then [11, Theorem 5.3] shows that $J(F[S]) = I(F, S, \xi)$. However, $I(F, S, \xi)$ is homogeneous only if ξ coincides with the equality relation. Therefore S is separative. Further, if F is a field of characteristic $p > 0$, then by [11, Theorem 5.3] $J(F[S]) = I(F, S, \xi_p)$. So S is p-separative for all p. It follows from [2, Theorem 4.16] that S is a semilattice B of cancellative semigroups S_b.

Now we will prove that S is cancellative. (It does not mean that B is a singleton.) Suppose the contrary: let there exist $x, y, z \in S$ such that $x \neq y$ and $xz = yz$. Then $x \in S_e$, $y \in S_f$, $z \in S_g$ for some $e, f, g \in B$.

If at least one of the elements e, f coincides with ef, then we may assume that $f = ef$, as the other case is analogous. If both e and f are not equal to ef, then setting $x' = x^2$, $y' = yx$, $f' = ef$ we get $x' \in S_e$, $y' \in S_{ef}$, $x' \neq y'$, $x'z = y'z$, $ef' = f'$ and therefore it is possible to substitute elements x', y', f' for x, y, f, respectively. Thus, without loss of generality we may assume that $f = ef$.

Further, we can replace z by $z' = zy$, because $xz' = yz'$. Since $z' \in S_{fg}$ and $e(fg) = f(fg) = fg$, to simplify the notation we assume that $eg = fg = g$ and there is no need of changing z. Consider the following two cases.

Case 1. $f \neq g$.

Let I denote the ideal generated in S by z. Set $T = S_e \cup S_f \cup I$. As in the proof of the 'if' part, S-homogeneity implies that J is T-homogeneous. Besides, T is separative but is not cancellative, since $x, y, z \in T$. Denote by M the ring of 2×2 matrices over a field F of characteristic zero. Let e_{ij}, where $i, j \in \{1, 2\}$, be the standard matrix with the identity element in the intersection of the i-row and j-column, all the others entries of which are zero. Put $N = Fe_{12}$, $U = S_e \cup S_f$. Clearly $e \geq f > g$ forces $U \cap I = \emptyset$. Consider the subring $R = N[U] + M[I]$ of the semigroup ring $M[S]$. Set $R_u = Nu$ for $u \in U$, and $R_i = Mi$ for $i \in I$. Then $R = \bigoplus_{t \in T} R_t$.

Consider the element $w = e_{1\,2}(x - y) \in N[U]$. For any $m \in M$, $i \in I$, there is $s \in S^1$ such that $i = sz$, and so $miw = me_{1\,2}s(zx - zy) = 0$. Therefore $M[I]w = 0$. Since $N[U]^2 = 0$, it follows that $Rw = 0$, whence $w \in J(R)$. By T-homogeneity $e_{1\,2}x \in J(R)$ implying $e_{1\,2}xz \in J(R)$. As $M[I]$ is an ideal of R, $e_{1\,2}xz \in J(M[I])$. However, [13, Theorem 4.6] shows that $M[I]$ is semisimple, giving a contradiction.

Case 2. $f = g$.

Then $xy, y^2 \in S_g$, $xyz = y^2z$ and therefore $xy = y^2$. Let I denote the ideal generated in S by y, and let $U = S_e$, $T = U \cup I$, $R = N[U] + M[I]$, $w = e_{1\,2}(x - y)$.

Take any $t \in T$, $r \in R_t$. If $t \in U$, then $r \in Nt$ and $N^2 = 0$ implies $rw = 0$. If $t \in I$, then $r = mt$ for some $m \in M$. By the definition of I there is $s \in S^1$ such that $t = sy$. Hence $Rw = 0$ and so $w \in J(R)$. Again J is T-homogeneous and we have $e_{1\,2}y \in J(M[I])$. This contradicts the semisimplicity of $M[I]$. Thus S is cancellative.

It is well known that each commutative cancellative semigroup S has a group of quotients G (cf. [2, § 1.10]). If G was not torsion-free, then G would contain an element w of period p for a prime p. This would give a contradiction with the p-cancellativeness of S, because $w = s^{-1}t$, $s, t \in S$ imply $s^p = t^p$. Thus S is embeddable in a torsion-free abelian group, as required. □

Note that a description of commutative semigroups S such that the Jacobson radical is homogeneous in every semigroup ring $R[S]$ follows from the results of [6]. For a con-commutative S, this problem still remains open.

Question. Let S be an arbitrary (not necessarily commutative) semigroup. Is it true that the Jacobson radical is S-homogeneous if and only if S is embeddable in a group G such that J is G-homogeneous?

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Department of Mathematics and Mechanics, Ural State University, Lenina 51, Ekaterinburg 620083, Russia

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