When $(E, \sigma(E, E'))$ is a DF-space?

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Abstract. Let (E, t) be a Hausdorff locally convex space. Either $(E, \sigma(E, E'))$ or $(E', \sigma(E', E))$ is a DF-space iff E is of finite dimension (THEOREM). This is the most general solution of the problem studied by Iyahen [2] and Radenovič [3].

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Throughout this paper, $E = (E, t)$ denotes a Hausdorff locally convex space and E' its topological dual. The classical Grothendieck's theorem asserts that $(E', \beta(E', E))$ is a complete DF-space (for definitions see below) provided that the topology t is metrizable ([1], [5, Ch. VI, Theorem 6.5, also p. 154]). The main result in this paper deals with the following natural question, first investigated in 1966 by Iyahen [2]: when $(E, \sigma(E, E'))$ (or $(E', \sigma(E', E))$) is a DF-space? Iyahen's result is that $(c_0, \sigma(c_0, l_1))$ is not a DF-space. In 1988, Radenovič [3] generalized this result and proved that if E is a Banach space of infinite dimension then neither $(E, \sigma(E, E'))$ nor $(E', \sigma(E', E))$ is a DF-space.

In this note we give the full answer to the question stated above (THEOREM). Our approach is most elementary (we work on dual pairs only) and the result is most general (no assumptions concerning the structure of E are required; compare [3, Theorems 1, 3 and Corollaries 1, 2]). We follow [4] and [5] for definitions and notations. For the convenience of the reader recall that E is said to be *countably quasibarrelled* if every strongly bounded subset of E' which is a countable union of t-equicontinuous subsets of E' is t-equicontinuous; if, in addition, E possesses a fundamental sequence of bounded sets then it is called a DF-space.

Theorem. *Let (E, E') be a dual pair. Then either $(E, \sigma(E, E'))$ or $(E', \sigma(E', E))$ is a DF-space iff E is of finite dimension.*

Proof (of the nontrivial case "only if"): Since both (E, E') and (E', E) is a dual pair, it is enough to consider the case $(E, \sigma(E, E'))$. Let $\{B_n\}$ be an arbitrary increasing fundamental sequence of bounded sets in $(E, \sigma(E, E'))$. Then $\{B_n^0\}$ is a base of neighbourhoods of zero in $(E', \beta(E', E))$. Let now $\{x_n' : n \in N\}$ be an arbitrary linearly independent sequence in E'. Then for every $n \in N$ there exists a real number $\alpha_n > 0$, such that $\alpha_n x_n' \in B_n^0$. Therefore $\alpha_n x_n' \to 0$ in $\beta(E', E)$; in particular, the sequence $\{\alpha_n x_n'\}$ is $\beta(E', E)$-bounded, and so it is

$\sigma(E, E')$-continuous. It follows that there is a finite subset K of E' with $\{\alpha_n x'_n : n \in N\} \subset (K^0)^0 \subset \operatorname{lin} K$; thus $\operatorname{lin}\{x'_n : n \in N\}$ is of finite dimension and, consequently, $\dim E < \infty$. □

The result stated in the corollary below indicates a large class of locally convex spaces which are not countably quasibarrelled. This problem was discussed in [3, p. 155, and Corollary 1] (for the Banach case only).

Corollary. *If E is a metrizable and barrelled space, then $(E', \sigma(E', E))$ is countably quasibarrelled iff E is of finite dimension.*

Proof: Since E' possesses a fundamental sequence of $\sigma(E', E)$-bounded sets, THEOREM is applied to complete the proof. □

References

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