On the regularity of the minimizer of a functional with exponential growth

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Abstract. Minimizers of a functional with exponential growth are shown to be smooth. The techniques developed for power growth are not applicable to the exponential case.

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In this note we prove that minimizers of the functional

$$I(u) = \int_\Omega \exp(|Du|^2)\,dx$$

over the class of all functions making $I(u)$ finite are smooth by showing that they are classical solutions of the corresponding Euler–Lagrange equation. If the exponential function is replaced by a power function, such a result is standard (see [3, Chapter 5]), but the techniques used do not extend to the present case.

Specifically we suppose that $v \in W^{1,1}(\Omega)$, for some open set $\Omega \in \mathbb{R}^n$, satisfies the inequality

$$\int_\Omega \exp(|Dv|^2)\,dx \le \int_\Omega \exp(|Dw|^2)\,dx \tag{1}$$

for all $w \in W^{1,1}(\Omega)$ with $w - v \in W_0^{1,1}(\Omega)$. (Obviously it suffices to consider only those w's making the right hand side of (1) finite.) Our main result is that v is a classical solution of the Euler–Lagrange equation corresponding to the functional I.

Theorem. *If $v \in W^{1,1}(\Omega)$ satisfies* (1) *for all $w \in W^{1,1}$ with $w - v \in W_0^{1,1}$ then v is a classical solution of the equation*

$$\{\delta^{ij} + 2D_ivD_jv\}D_{ij}v = 0 \quad \text{in } \Omega. \tag{2}$$

Since v is a classical solution of (2), it follows that $v \in C^2(\Omega)$ and then the usual linear theory shows that v is locally analytic.

The question of the smoothness of v was posed to the author by Prof. Mariano Giaquinta at the Banach Center for Mathematics. The author is grateful to Prof. Giaquinta for posing the question and to the Center for providing the opportunity to meet Prof. Giaquinta.

In fact, a slightly different question was posed, one originally asked by J. Eells, who wanted to know if v is a weak solution of the Euler–Lagrange equation

$$\operatorname{div}(\exp(|Dv|^2)\, Dv) = 0 \quad \text{in } \Omega. \tag{2$'$}$$

Because classical solutions are weak solutions, our theorem gives an affirmative answer to Eells's question. Further details on this question, including alternative methods for showing that v is a weak solution, can be found in [1].

1. Proof of the Theorem.

To prove our theorem, we show that, for all balls $B = B(x_0, R) \subset \Omega$, v agrees with the classical solution of

$$\{\delta^{ij} + 2D_iuD_ju\}D_{ij}u = 0 \ \text{ in } B, \quad u = v \ \text{ on } \partial B. \tag{3}$$

The existence of u can be inferred from the remarks in [2, Section 15.6] and [2, Theorem 15.14]; its existence is also an easy consequence of our approximate scheme.

To begin, we set $\varepsilon_0 = \operatorname{dist}(B, \partial\Omega)$, and we fix a nonnegative C^∞ function φ supported in the unit ball of $\mathbb{R}^n$ with $\int \varphi(z)\, dz = 1$. For $\varepsilon \in (0, \varepsilon_0)$, we define v_ε by

$$v_\varepsilon(x) = \int_{\mathbb{R}^n} v(x + \varepsilon z)\varphi(z)\, dz.$$

(Note that $0 < \varepsilon < \varepsilon_0$ guarantees that $x + \varepsilon z \in \Omega$ for $z \in \operatorname{supp}\varphi$.) Then for any convex nonnegative, increasing function G we have

$$\begin{aligned}
\int_B G(|Dv_\varepsilon|)\, dx &\le \int_{\mathbb{R}^n}\int_B G(|Dv(x + \varepsilon z)|)\, dx\, \varphi(z)\, dz \\
&\le \sup_{|z|<1} \int_B G(|Dv(x + \varepsilon z)|)\, dx \\
&\le \int_{B(x_0, R+\varepsilon)} G(|Dv|)\, dx
\end{aligned}$$

by using Jensen's inequality and Tonelli's theorem.

Now, let u_ε be the $C^2(\overline{B})$ solution of

$$\{\delta^{ij} + D_iu_\varepsilon D_ju_\varepsilon\}D_{ij}u_\varepsilon = 0 \ \text{ in } B, \quad u_\varepsilon = v_\varepsilon \ \text{ on } \partial B \tag{4}$$

given by [2, Theorem 11.5]. Because $\sup_B |v_\varepsilon| \le \sup_\Omega |v|$, the maximum principle gives a bound on u_ε which is independent of ε. It then follows from [7] (see [4, pp. 62–63] for details) or from Example 2 on p. 585 of [6] that for any compact subset K of B there is a uniform bound on $\sup_K |Du_\varepsilon|$ independent of ε. Classical elliptic theory then gives uniform local bounds on all higher derivatives of u_ε. Now we use the weak form of (4), namely

$$\int \exp(|Du_\varepsilon|^2)Du_\varepsilon \cdot D\psi\, dx = 0 \tag{4$'$}$$

for all $\psi \in C_0^1(\Omega)$.

The convexity of $\exp(t)$ implies that

$$\exp(|Du_\varepsilon|^2) \le \exp(|Dv_\varepsilon|^2) + 2\exp(|Du_\varepsilon|^2) Du_\varepsilon \cdot D(u_\varepsilon - v_\varepsilon);$$

integrating this inequality over B and using $(4)'$ with $\psi = u_\varepsilon - v_\varepsilon$ yields

$$\int_B \exp(|Du_\varepsilon|^2)\,dx \le \int_B \exp(|Dv_\varepsilon|^2)\,dx \le \int_{B(x_0,R+\varepsilon)} \exp(|Dv_\varepsilon|^2)\,dx.$$

Therefore $\int_B |Du_\varepsilon|^{n+1}\,dx$ is uniformly bounded, so the Sobolev–Morrey imbedding theorem and the Arzela–Ascoli theorem give a sequence $(\varepsilon(m))$ with $\varepsilon(m) \to 0$ and a function $u \in C^0(\overline{B})$ such that $u_{\varepsilon(m)} \to u$ uniformly in B. The uniform local bounds on derivatives of the u_ε's imply that $(D^2 u_{\varepsilon(m)})$ converges uniformly on compact subsets of B, and hence $u \in C^2(B)$ and u solves (3). Fatou's lemma gives

$$\int_B \exp(|Du|^2)\,dx \le \int_B \exp(|Dv|^2)\,dx$$

and the uniform convexity of the map E, defined by $E(p) = \exp(|p|^2)$, implies that $Du = Dv$ a.e. Since u and v are continuous with $u = v$ on ∂B, it follows that $u \equiv v$, which proves the theorem.

2. A generalization.

In fact, the special form of the functional I is not important to the underlying argument. This form is only used to obtain appropriate uniform estimates. Let us suppose that I is given by

$$I(u) = \int_\Omega f(x,u,Du)\,dx$$

and that F satisfies the following conditions

(F1) $F(x,z,p)$ is convex in (z,p) and strictly convex in p,
(F2) $F(x,z,p) \ge 0$, $F \in C^3(\Omega \times \mathbb{R} \times \mathbb{R}^n)$

for all $(x,z,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$. Also write Q for the Euler–Lagrange operator associated with $I : Qu = \operatorname{div} F_p(x,u,Du) - F_z(x,u,Du)$ and suppose that

(Q1) The Dirichlet problem $Qu = 0$ in $B(R)$, $u = w$ on $\partial B(R)$ is solvable in $C^3\left(\overline{B(R)}\right)$ for any $w \in C^3(\partial B(R))$,

(note that u is unique because of (F1) and the maximum principle),

(Q2) the $C^2(K)$ norm of u can be estimated in terms of $|w|_{L^\infty}$ for any compact subset K of $B(R)$,
(Q3) a modulus of continuity for u can be estimated in terms of the modulus of continuity of w.

Structure conditions on F which guarantee (Q1), (Q2), (Q3) can be found in [2], [3], [5], [6]. Suppose also that there is a sequence (w_m) in $C^3(\overline{B})$, which converges uniformly to v, such that $I(w_m) \to I(v)$. If u_m is the $C^3(\overline{B})$ solution of $Qu_m = 0$ in B, $u_m = w_m$ on ∂B, convexity gives

$$\int_B F(x, u_m, Du_m)\,dx \le \int_B F(x, w_m, Dw_m)\,dx. \tag{5}$$

The uniform estimates in (Q2) and (Q3) allow us to extract a convergent subsequence which converges to a classical solution of $Qu = 0$ in B, $u = v$ on ∂B. Finally (5), and Fatou's lemma imply that

$$\int_B F(x, u, Du)\,dx \le \int_B F(x, v, Dv)\,dx,$$

so strict convexity again gives $u \equiv v$.

The considerations of this section apply also to minimization problems on manifolds, in particular the problem dealt with in [1]. In a coordinate neighborhood, the functional F can be written as

$$F(x, z, p) = \exp\Big(\sum_{i,j=1}^{n} g^{ij}(x) p_i p_j\Big)\, g(x)$$

for smooth functions g^{ij} and g such that (g^{ij}) is a positive definite matrix and g is a positive scalar, and $x \in \Omega$, some open subset of $\mathbb{R}^n$. If B is a ball whose closure lies in Ω, the conditions (F1) and (F2) are clear. Moreover the only nonstandard element in (Q1), (Q2), (Q3) is the gradient estimate, which is proved by rewriting the Euler-Lagrange equation as

$$\sum_{i,j=1}^{n} D_i\Big(\exp\Big(\sum_{k,m=1}^{n} g^{km}(x) D_k u D_m u\Big) g^{ij}(x) D_j u\Big) +$$
$$\frac{1}{g(x)} \sum_{i,j=1}^{n} g^{ij}(x) D_i u D_j g(x) \Big(\exp\Big(\sum_{k,m=1}^{n} g^{km}(x) D_k u D_m u\Big)\Big) = 0$$

and applying the results of [5].

Only a small effort is needed to construct the sequence (w_m). For each integer m, choose $\varepsilon(m) > 0$ so that $\overline{B(R + \varepsilon(m))} \subset \Omega$,

$$\begin{aligned} |g(x) - g(y)| &\le \frac{1}{m} g(x), \\ \max_{i,j} |g^{ij}(x) - g^{ij}(y)| &\le \frac{1}{m^4} \min_{\xi=1} \sum_{i,j=1}^{n} g^{ij}(x)\xi_i\xi_j \end{aligned} \tag{6}$$

for all x, y in $B(R + \varepsilon(m))$ such that $|x - y| < \varepsilon(m)$. (The positivity and smoothness of (g^{ij}) and g guarantee such an $\varepsilon(m)$.) Choosing $w_m = (1 - \frac{1}{m}) v_{\varepsilon(m)}$, we infer from the vector version of Jensen's inequality, along with Tonelli's theorem and (6), that

$$\int_B F(x, w_m, Dw_m)\, dx \leq \left(1 + \frac{1}{m}\right) \int_{B(R+\varepsilon(m))} F(x, v, Dv)\, dx.$$

The general properties of mollification imply that $w_m \in C^3(\overline{B})$ and that the modulus of continuity of w_m can be estimated uniformly in m.

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