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On stabbing triangles by lines in 3-space

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Abstract. We give an example of a set P of $3n$ points in ℝ3 such that, for any partition of P into triples, there exists a line stabbing $\Omega(\sqrt{n})$ of the triangles determined by the triples.

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We begin with a definition. Let T be a set of points in $\mathbb{R}^d$ and let f be a flat. We say that f *crosses* T if f transversally intersects the relative interior of the convex hull of T. In particular, a line crosses a triple of non-collinear points in $\mathbb{R}^3$ if it intersects the triangle determined by the triple in a single interior point.

Given a set P of $2n$ points in the plane, one can partition P into n pairs in such a way that no line crosses more than $C\sqrt{n}$ of the pairs, where C is an absolute constant. It is easy to construct point sets for which this bound is best possible, up to the value of the constant. This result was proved by Chazelle and Welzl [CW89], and it found numerous applications in the design of geometric algorithms (see e.g. [Aga90]) as well as in combinatorial geometry [Pac91] and discrepancy theory [MWW93]. It has been generalized in various directions.

One possible generalization to higher dimension is to consider a set P of $2n$ points in $\mathbb{R}^d$ and to ask for a partition of P into pairs such that the maximum number of pairs crossed by any single hyperplane is as small as possible. In dimension d, a tight bound on this crossing number is $\Theta(n^{1-1/d})$ [CW89]. Another generalization arising naturally in various applications and posed explicitly as an open problem by Welzl [Wel92] is the following:

> *Let $\mathcal{T}$ be a partition of a set P of $3n$ points in $\mathbb{R}^3$ into triples, and denote by $\kappa(\mathcal{T})$ the maximum number of triples of $\mathcal{T}$ which can be simultaneously crossed by a single line. Put*

$$\kappa(n) = \max_P \min_{\mathcal{T}} \kappa(\mathcal{T}),$$

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where the maximum is taken over all sets P of $3n$ points in $\mathbb{R}^3$ and the minimum over all partitions of P into triples. Find the order of magnitude of $\kappa(n)$.

It is known (although, to our knowledge, it has not appeared in print) that $\kappa(n) = O(\sqrt{n})$; this can be established, e.g. by projecting P orthogonally onto a general plane ρ, and by partitioning the projected set $\bar{P}$ into triples such that any line within ρ crosses $O(\sqrt{n})$ of the triples. Such a partition within ρ is possible by a result of [Mat92], which generalizes the aforementioned Chazelle-Welzl result in yet another direction. It is not difficult to check that by 'lifting' this partition back to P, we obtain the desired partition. In this note we show that this result is asymptotically tight:

Proposition. $\kappa(n) = \Omega(\sqrt{n})$.

Proof: Let Γ denote the hyperbolic paraboloid with equation $z = xy$ in $\mathbb{R}^3$. This surface plays a key role in our example as well as in many other constructions related to the geometry of lines in 3-space, see e.g. [CEGS89] [CP90].

Without loss of generality, we assume that $3n$ is of the form m^2, and we define an auxiliary point set

$$P = \{(i, j, ij);\ i, j = 1, 2, \ldots, m\} \subset \Gamma\,.$$

Our example, a point set $\tilde{P}$ witnessing $\kappa(n) = \Omega(\sqrt{n})$, arises by a small perturbation of P. We choose a small enough number $\varepsilon > 0$ and put

$$\tilde{P} = \{(i + \varepsilon j, j + \varepsilon i, (i + \varepsilon j)(j + \varepsilon i));\ i, j = 1, 2, \ldots, m\} \subset \Gamma\,.$$

The only lines intersecting the quadric surface Γ in more than 2 points are parallel to the yz- or xz-planes. From this it is easily seen that no 3 points of $\tilde{P}$ are collinear.

Let $\mathcal{T}$ be a partition of $\tilde{P}$ into triples. Consider a triple $\tilde{T} = \{\tilde{a}, \tilde{b}, \tilde{c}\} \in \mathcal{T}$, let $T = \{a, b, c\}$ be the corresponding triple of points of the 'unperturbed' set P, and let $\bar{T}$ be the vertical projection of the triangle $\triangle abc$ onto the xy-plane. We say that the triple $\tilde{T}$ is of one of three types, as follows:

I if none of the sides of $\bar{T}$ are parallel to the x- or y-axes,
II if at least one but not all sides of $\bar{T}$ are parallel to the axes, and
III if the entire $\bar{T}$ is a segment parallel to the x-axis or y-axis.

It can be checked that this classification exhausts all possibilities. We analyze three cases according to which type has the majority of triples.

Case I (at least $n/3$ triples are of type I): We show that there is a line crossing $\Omega(\sqrt{n})$ of the triples of the form T, with $\tilde{T} \in \mathcal{T}$ (that is, the triangles formed by the unperturbed points). If $\varepsilon > 0$ is chosen small enough, this line also crosses the corresponding triples after the perturbation.

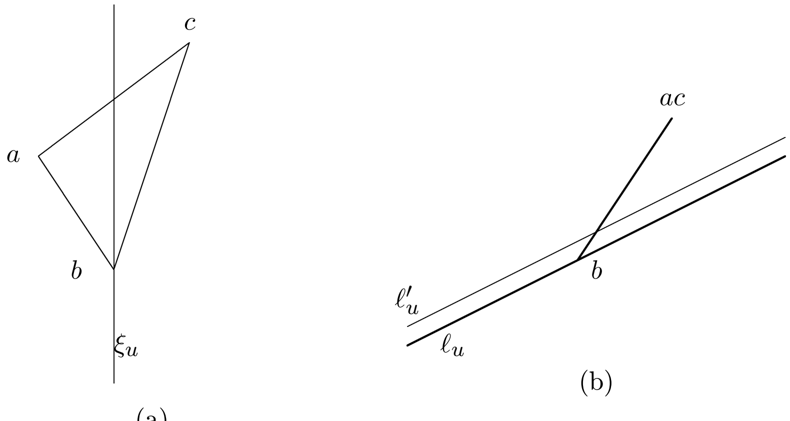

Figure 1: Type I triple: (a) vertical projection, (b) situation within the plane ξ_u.

Consider a triple $\tilde{T}$ of type I (see Fig. 1). The points a, b, c, have distinct x-coordinates; let b be the one with the middle x-coordinate, and let this coordinate $x(b)$ be equal to u. We consider the situation in the plane $\xi_u = \{x = u\}$. The surface Γ intersects ξ_u in the line $\ell_u = \{x = u, z = uy\}$. While the point b lies on ℓ_u, the intersection of the segment ac with ξ_u has a positive distance from ℓ_u (as the line ac only intersects the surface Γ at the two points a and c). Hence, any line in ξ_u parallel to ℓ_u, lying on the appropriate side of and sufficiently close to it crosses the triangle abc.

For each $i = 1, 2, \ldots, m$, choose one line $\ell'_i \subset \xi_i$ parallel to ℓ_i, lying below ℓ_i and very close to it, and similarly ℓ''_i above ℓ_i. By the above considerations, each of the at least $n/3$ triples of type I is crossed by at least one line of the form ℓ'_i or ℓ''_i. Hence there is a line crossing at least $(n/3)/2m = \Omega(\sqrt{n})$ triples of type I.

Case II (at least $n/3$ triples are of type II): We again carry out the argument with the unperturbed points. Let $\tilde{T}$ be a type II triple, let a, b, c be the corresponding points of P, and let the side ab be parallel to the xz-plane (the other case, ab parallel to the yz-plane, is handled symmetrically), let d be the midpoint of the segment ab, set $u = x(d)$, and consider the situation in the vertical plane $\xi_u = \{x = u\}$. The triangle abc intersects ξ_u in a segment whose one endpoint d lies on the surface Γ while the other endpoint does not lie on Γ. Thus, as in the case I, any line in ξ_u parallel to the line $\ell_u = \xi_u \cap \Gamma$, lying on appropriate side of ℓ_u and close enough to it crosses the triple $\{a, b, c\}$. As the x-coordinate u of the midpoint of ab has fewer than $2m$ distinct values for all triples, we can choose fewer than $8m$ lines (including the lines for the symmetric case with ab parallel to the yz-plane) so that every type II triple is crossed by at least one of them. Thus one of these lines crosses $\Omega(\sqrt{n})$ triples.

Case III (at least $n/3$ triples are of type III): Let $\tilde{T}$ be a type III triple, suppose that its triangle is almost parallel to the xz-plane; the other case is handled symmetrically. Let the points of $\tilde{T}$ be

$$\begin{aligned}\tilde{a} &= (u + \varepsilon j, j + \varepsilon u, (u + \varepsilon j)(j + \varepsilon u)),\\ \tilde{b} &= (v + \varepsilon j, j + \varepsilon v, (v + \varepsilon j)(j + \varepsilon v)),\\ \tilde{c} &= (w + \varepsilon j, j + \varepsilon w, (w + \varepsilon j)(j + \varepsilon w)),\end{aligned}$$

with $u < v < w$. We consider the situation within the plane $\xi_v = \{x = v\}$. We calculate the intersections d, e of the segments ab and ac, respectively, with the plane ξ_v, and we find their vertical distance to the line $\ell_v = \xi_v \cap \Gamma$. It turns out that the vertical distance of d from ℓ_v is $\varepsilon^2 j(v-u) - \varepsilon^3 j$; the important fact for us that it is $O(\varepsilon^2)$ (with n fixed and $\varepsilon \to 0$) — this can also be seen by geometric considerations. On the other hand, the vertical distance of the point e from ℓ_v is $\varepsilon(w-v)(v-u) + O(\varepsilon^2) = \Omega(\varepsilon)$ as u, v, w are all distinct. Hence, if we choose lines $\ell'_v, \ell''_v \subset \xi_v$ parallel to ℓ_v, lying above and below it at a vertical distance $\varepsilon^{3/2}$ (say), then one of them intersects the segment de and thus it crosses the triple $\tilde{T}$, provided that $\varepsilon > 0$ is small enough. The argument is concluded as in cases I and II, finishing the proof of the proposition. □

Remarks. (1) One might object that the above construction does not produce a set of points "in general position", as the entire point set is contained in the algebraic surface Γ. This objection can be addressed as follows: Start with the set $\tilde{P}$ as above. For every partition $\mathcal{T}$ of points of $\tilde{P}$ into triples, we can fix a line witnessing the crossing number of the partition. This line transversally intersects the interior of each of the crossed triangles, and hence if the vertices are moved by a sufficiently small amount, the crossing still exists. Taking the minimum of these amounts over all witnessing lines and all crossed triangles, we obtain a distance δ such that any perturbation of $\tilde{P}$ moving each point by at most δ preserves the property that any partition of the point set into triples has a line crossing $\Omega(n^{1/2})$ triples.

(2) The same proof, with little modification, can be used to show the following more general result (the same point set will do):

> *For any $n > 0$, there is a set of n points in $\mathbb{R}^3$ such that any collection of m triangles (not necessarily distinct) spanned by these points has a line crossing, in the above sense, $\Omega(m/n^{1/2})$ triangles.*

In particular, we have

Corollary. *For any $n > 0$, there exists a set S_n of n points in $\mathbb{R}^3$ such that for any triangulation of the convex hull of S_n there is a line meeting $\Omega(n^{1/2})$ simplices of the triangulation.*

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