Notes on approximation in the Musielak-Orlicz sequence spaces of multifunctions

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Abstract. We introduced the notion of $(\mathbf{X}, \mathrm{dist}, \mathcal{V})$-boundedness of a filtered family of operators in the Musielak-Orlicz sequence space X_φ of multifunctions. This notion is used to get the convergence theorems for the families of $\mathbf{X}$-linear operators, $\mathbf{X}$-dist-sublinear operators and $\mathbf{X}$-dist-convex operators. Also, we prove that X_φ is complete.

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1. Introduction

Let $\mathbf{N}$ be the set of all nonnegative integers. Let l^φ be the Musielak-Orlicz sequence space generated by a modular

$$\varrho(x) = \sum_{i=o}^{\infty} \varphi_i(t_i), x = (t_i),$$

where $\varphi = (\varphi_i)$ is a sequence of φ-functions with parameter, i.e. for every $i \in \mathbf{N}$ we have: $\varphi_i : R \to R_+ = [0, \infty)$, $\varphi_i(u)$ is an even continuous function, equal to zero iff $u = 0$ and nondecreasing for $u \geq 0$, $\lim_{u\to\infty} \varphi_i(u) = \infty$. Let

$$X = \{F : \mathbf{N} \to 2^R : F(i) \ \text{ is nonempty and compact for every } \ i \in \mathbf{N}\}.$$

Every function from $\mathbf{N}$ to 2^R we will be called multifunction. For every $F \in X$ we define the functions $\underline{f}(F)$ and $\overline{f}(F)$ by the formulas:

$$\underline{f}(F)(i) = \min_{x\in F(i)} x, \ \overline{f}(F)(i) = \max_{x\in F(i)} x \ \text{ for every } \ i \in \mathbf{N}.$$

Let now $[a, b]$ denote a compact segment for all $a, b \in R$, $a \leq b$. Define

$$X_\varphi = \{F \in X : \underline{f}(F), \overline{f}(F) \in l^\varphi\},$$

$$\tilde{X}'_\varphi = \{F \in X_\varphi : F(i) = \bigcup_{k=1}^{n_i} [a_k(i), b_k(i)] \ \text{ for every } \ i \in \mathbf{N}, \ \text{ where } \ n_i \in \mathbf{N}\backslash\{0\},$$
$$a_k(i), b_k(i) \in R \ \text{ for } \ i \in \mathbf{N}, k = 1, \dots, n_i\}.$$

Let $\mathbf{V}$ be an abstract set of indices. Let $\mathcal{V}$ be a filter of subsets of $\mathbf{V}$. Let $\mathbf{0} : \mathbf{N} \to R$ be such that $\mathbf{0}(i) = 0$ for every $i \in \mathbf{N}$.

In [6] a general approximation theorem in modular spaces was obtained for linear operators. This theorem was extended in [1] and [7] to some nonlinear operators in $L^\varphi(\Omega, \Sigma, \mu)$, in [2] to $\tilde{X}_\varphi$-linear operators in $\tilde{X}_\varphi$, in [3] to some operators in $\tilde{X}_\varphi$ and in [5] to some operators in $X_{d,\varphi}$. The space X_φ was introduced in [4] without studying its completeness. The aim of this note is to prove that X_φ is complete and to obtain an extension of the results of [2], [3] to the case of approximation by some families of operators in the sequence spaces of multifunctions $\tilde{X}'_\varphi$ and X_φ.

2. General theorems

Definition 1. Let $A, B \subset R$ be nonempty and compact. We introduce the Hausdorff metric by the formula:

$$\operatorname{dist}(A,B) = \max(\max_{x\in A} \min_{y \in B} \mid x-y \mid,\ \max_{y\in B} \min_{x\in A} \mid x-y \mid).$$

Theorem 1. *Let $F_n \in X_\varphi$ for every $n \in \mathbf{N}$. If for every $\varepsilon > 0$ and every $a > 0$ there is $K > 0$ such that $\varrho(a \operatorname{dist}(F_n(\cdot), F_m(\cdot))) < \varepsilon$ for all $m, n > K$, then there exists $F \in X_\varphi$, such that $\varrho(a \operatorname{dist}(F_n(\cdot), F(\cdot))) \to 0$ as $n \to \infty$ for every $a > 0$.*

PROOF: Let the sequence $\{F_n\}$ fulfil the assumptions of the Theorem 1. So $\{F_n(i)\}$ is a Cauchy sequence for every $i \in \mathbf{N}$ in the complete space of all compact nonempty subsets of R with Hausdorff metric. Hence there are compact nonempty $F_i \subset R$ such that $\operatorname{dist}(F_n(i), F_i) \to 0$ as $n \to \infty$ for every $i \in \mathbf{N}$. Let $F(i) = F_i$ for every $i \in \mathbf{N}$. Applying the Fatou lemma we easily obtain that $\varrho(a \operatorname{dist}(F_n(\cdot), F(\cdot))) \le \varepsilon$ for every $n > K$. Also we have for every $a > 0$ and g equal $\underline{f}(F)$ or $\overline{f}(F)$

$$\begin{aligned}\varrho(ag) &\le \varrho(a \operatorname{dist}(F(\cdot), 0))\\ &\le \varrho(2a \operatorname{dist}(F_n(\cdot), F(\cdot))) + \varrho(2a \operatorname{dist}(F_n(\cdot), 0))\\ &\le \varrho(2a \operatorname{dist}(F_n(\cdot), F(\cdot))) + \varrho(4a\underline{f}(F_n)) + \varrho(4a\overline{f}(F_n)).\end{aligned}$$

So $\underline{f}(F), \overline{f}(F) \in l^\varphi$. □

The space X_φ will be called Musielak-Orlicz sequence space of multifunctions.

Definition 2. A function $g : \mathbf{V} \to R$ tends to zero with respect to $\mathcal{V}$, written $g(v) \xrightarrow{\mathcal{V}} 0$, if for every $\varepsilon > 0$ there is $V \in \mathcal{V}$ such that $\mid g(v) \mid < \varepsilon$ for every $v \in V$.

Let now $\mathbf{X}$ be equal to $\tilde{X}'_\varphi$ or to X_φ.

Definition 3. An operator $A : \mathbf{X} \to \mathbf{X}$ will be called an $\mathbf{X}$-dist-sublinear operator, if for all $F, G \in \mathbf{X}$ and $a, b \in R$

$$\operatorname{dist}(A(aF + bG)(i), 0) \le \mid a \mid \operatorname{dist}(A(F)(i), 0) + \mid b \mid \operatorname{dist}(A(G)(i), 0)$$

for every $i \in \mathbf{N}$.

Definition 4. An operator $B : \mathbf{X} \to \mathbf{X}$ will be called an $\mathbf{X}$-dist-convex operator, if for all $F, G \in \mathbf{X}$, $a, b \geq 0$, $a + b = 1$,

$$\operatorname{dist}(B(aF + bG)(i), (aF + bG)(i))$$
$$\leq a \operatorname{dist}(B(F)(i), F(i)) + b \operatorname{dist}(B(G)(i), G(i)) \quad \text{for every} \quad i \in \mathbf{N}.$$

Definition 5. An operator $C : \mathbf{X} \to \mathbf{X}$ will be called an $\mathbf{X}$-linear operator if for all $F, G \in \mathbf{X}$, $a, b \in R$,

$$C(aF + bG)(i) = aC(F)(i) + bC(G)(i) \quad \text{for every} \quad i \in \mathbf{N}.$$

Remark 1. If A is $\mathbf{X}$-linear operator, then it is $\mathbf{X}$-dist-sublinear operator and $\mathbf{X}$-dist-convex operator.

Definition 6. A family $T = (T_v)_{v \in \mathbf{V}}$ of operators $T_v : \mathbf{X} \to \mathbf{X}$, for every $v \in \mathbf{V}$ will be called $(\mathbf{X}, \operatorname{dist}, \mathcal{V})$-bounded, if there exist constants $k_1, k_2 > 0$ and a function $g : \mathbf{V} \to R_+$ such that $g(v) \xrightarrow{\mathcal{V}} 0$, and for all $F, G \in \mathbf{X}$ there is a set $V_{F,G} \in \mathcal{V}$ for which

$$\varrho(a \operatorname{dist}(T_v(F)(\cdot), T_v(G)(\cdot))) \leq k_1 \varrho(a k_2 \operatorname{dist}(F(\cdot), G(\cdot))) + g(v)$$

for all $v \in V_{F,G}$ and every $a > 0$.

Definition 7. Let $F_v \in X_\varphi$ for every $v \in \mathbf{V}$. Let $F \in X_\varphi$. We write $F_v \xrightarrow{d,\varphi,\mathcal{V}} F$, if for every $\varepsilon > 0$ and every $a > 0$ there exists $V \in \mathcal{V}$ such that $\varrho(a \operatorname{dist}(F_v(\cdot), F(\cdot))) < \varepsilon$ for every $v \in V$.

Remark 2. If $F, G \in X_\varphi$, then $\operatorname{dist}(F(\cdot), G(\cdot)) \in l^\varphi$.

Definition 8. Let $S \subset \mathbf{X}$.

$$S_{\mathbf{X},d,\varphi,\mathcal{V}} = \{F \in \mathbf{X} : F_v \xrightarrow{d,\varphi,\mathcal{V}} F, \quad \text{for some} \quad F_v \in S, v \in \mathbf{V}\}.$$

Theorem 2. *Let the family $T = (T_v)_{v \in \mathbf{V}}$ of $\mathbf{X}$-dist-sublinear operators for every $v \in \mathbf{V}$, be $(\mathbf{X}, \operatorname{dist}, \mathcal{V})$-bounded. Let $S_o \subset \mathbf{X}$ and let $T_v(F) \xrightarrow{d,\varphi,\mathcal{V}} \mathbf{0}$ for every $F \in S_o$. Let S be the set of all finite linear combinations of elements of the set S_o. Then $T_v(F) \xrightarrow{d,\varphi,\mathcal{V}} \mathbf{0}$ for every $F \in S_{\mathbf{X},d,\varphi,\mathcal{V}}$.*

The proof analogous to that of Theorem 1 in [3] is omitted.

Theorem 3. *Let the family $T = (T_v)_{v \in \mathbf{V}}$ of $\mathbf{X}$-dist-convex operators for every $v \in \mathbf{V}$ be $(\mathbf{X}, \operatorname{dist}, \mathcal{V})$-bounded. Let $S_o \subset \mathbf{X}$ and let $T_v(F) \xrightarrow{d,\varphi,\mathcal{V}} F$ for every $F \in S_o$. Let now S be the set of all finite convex combinations of elements of the set S_o. Then $T_v(F) \xrightarrow{d,\varphi,\mathcal{V}} F$ for every $F \in S_{\mathbf{X},d,\varphi,\mathcal{V}}$.*

The proof analogous to that of Theorem 2 in [3] is omitted.

Theorem 4. *Let the family $T = (T_v)_{v \in \mathbf{V}}$ of $\mathbf{X}$-linear operators for every $v \in \mathbf{V}$, be $(\mathbf{X}, \operatorname{dist}, \mathcal{V})$-bounded. Let $S_o \subset \mathbf{X}$ and let $T_v(F) \xrightarrow{d,\varphi,\mathcal{V}} F$ for every $F \in S_o$. Let now S be the set of all finite linear combinations of elements of the set S_o. Then $T_v(F) \xrightarrow{d,\varphi,\mathcal{V}} F$ for every $F \in S_{\mathbf{X},d,\varphi,\mathcal{V}}$.*

The proof analogous to that of Theorem 1 in [2] is omitted.

3. Applications

Let now $\mathbf{V} = \mathbf{N}$ and the filter $\mathcal{V}$ will consist of all sets $V \subset \mathbf{V}$ which are complements of finite sets.

We shall say that φ is τ_+-bounded, if there are constants $k_1, k_2 \geq 1$ and a double sequence $\{\varepsilon_{n,j}\}$ such that

$$\varphi_{n+j}(u) \leq k_1 \varphi_n(k_2 u) + \varepsilon_{n,j}$$

for $u \in R$, $n, j = 0, 1, \dots$, where $\varepsilon_{n,j} \geq 0$, $\varepsilon_{n,0} = 0$, $\varepsilon_j = \sum\limits_{n=0}^{\infty} \varepsilon_{n,j} \to 0$ as $j \to \infty$, $\mathbf{s} = \sup_{j \in \mathbf{N}} \varepsilon_j < \infty$. Let $K_{v,j} : \mathbf{V} \times \mathbf{V} \to R_+$ and let the family $(K_v)_{v \in \mathbf{V}}$ be almost-singular, i.e. $\sigma(v) = \sum_{j=0}^{\infty} K_{v,j} \leq \sigma < \infty$ for all $v \in \mathbf{V}$ and $\frac{K_{v,j}}{\sigma(v)} \xrightarrow{\mathcal{V}} 0$ for $j = 1, 2, \dots$. Let $F \in X_\varphi$. We define a family $\mathcal{T} = (\mathcal{T}_v)_{v \in \mathbf{V}}$ of operators by the formula:

$$\mathcal{T}_v(F)(i) = \sum_{j=0}^{i} K_{v,i-j} F(j) \quad \text{for every} \quad i \in \mathbf{V}.$$

Lemma 1. *Let $(K_v)_{v \in \mathbf{V}}$ be almost-singular, let $\varphi = (\varphi_i)_{i \in \mathbf{V}}$ be τ_+-bounded and φ_i be convex for every $i \in \mathbf{V}$, then $\mathcal{T}_v : l^\varphi \to l^\varphi$ for every $v \in \mathbf{V}$.*

The proof analogous to that of Proposition 4 in [6] is omitted.

Lemma 2. *If the assumptions of Lemma 1 hold, then the family $\mathcal{T} = (\mathcal{T}_v)_{v \in \mathbf{V}}$ is $(X_\varphi, \mathrm{dist}, \mathcal{V})$-bounded and $\mathcal{T}_v$ is X_φ-linear-operator for every $v \in \mathbf{V}$.*

PROOF: From Lemma 1 we easily obtain that $\mathcal{T}_v : X_\varphi \to X_\varphi$. We prove that $\mathcal{T}$ is $(X_\varphi, \mathrm{dist}, \mathcal{V})$-bounded family of X_φ-linear operators. Let $a, b \in R$, $F, G \in X_\varphi$, $i \in \mathbf{V}$. We have

$$\begin{aligned} \mathcal{T}_v(aF + bg)(i) &= \sum_{j=0}^{i} K_{v,i-j}(aF(j) + bG(j)) \\ &= a \sum_{j=0}^{i} K_{v,i-j} F(j) + b \sum_{j=0}^{i} K_{v,i-j} G(j) \\ &= a\mathcal{T}_v(F)(i) + b\mathcal{T}_v(G)(i), \\ &\varrho(a\,\mathrm{dist}(\mathcal{T}_v(F)(\cdot), \mathcal{T}_v(G)(\cdot))) \end{aligned}$$

$$\leq \sum_{i=0}^{\infty} \varphi_i(a \sum_{j=0}^{i} K_{v,i-j} \operatorname{dist}(F(j), G(j)))$$
$$\leq k_1 \varrho(ak_2 \sigma \operatorname{dist}(F(\cdot), G(\cdot))) + c(v),$$

where $c(v) = \frac{1}{\sigma(v)} \sum_{i=1}^{\infty} K_{v,i} \varepsilon_i \xrightarrow{\mathcal{V}} 0$ (see [6], p. 109, the proof of Proposition 4). $\square$

We easily obtain (see [7], 8.13 and 8.14) the following

Lemma 3. *Let $\varphi = (\varphi_i)_{i=0}^{\infty}$ satisfy the condition (δ_2). Let $F \in X_\varphi$ and $F = (F(i))_{i=0}^{\infty}$. Let F_v be such that $F_v(i) = F(i)$ for $i = 0, 1, \ldots, v$ and $F_v(i) = 0$ for $i > v$. Then $F_v \xrightarrow{d,\varphi,\mathcal{V}} F$.*

Remark 3. If $A \subset R$ is nonempty and compact and $a \in R$, then

$$\operatorname{dist}(aA, A) \leq \mid 1 - a \mid \max_{x \in A} \mid x \mid .$$

Proof: Let $A \subset R$ be nonempty and compact and let $a \in R$, we have

$$\operatorname{dist}(aA, A) = \max(\max_{x \in aA} \min_{y \in A} \mid x - y \mid,\ \max_{y \in A} \min_{x \in aA} \mid x - y \mid)$$
$$= \max(\max_{z \in A} \min_{y \in A} \mid az - y \mid,\ \max_{y \in A} \min_{z \in A} \mid az - y \mid) \leq \mid 1 - a \mid \max_{x \in A} \mid x \mid .$$

Now, let us denote: $x_{j,K_v} = \{\underbrace{0, \ldots, 0}_{j-times}, K_{v,1}, K_{v,2}, \ldots\}$. $\square$

Theorem 5. *Let the assumptions of Lemmas 1 and 3 hold. If $x_{j,K_v} \xrightarrow{d,\varphi,\mathcal{V}} \mathbf{0}$ for every $j \in \mathbf{V}$, $K_{v,o} \xrightarrow{\mathcal{V}} 1$, then $\mathcal{T}_v(F) \xrightarrow{d,\varphi,\mathcal{V}} F$ for every $F \in X_\varphi$.*

Proof: Let us denote:

$E_k(A) = (\Delta_{i,k}(A))_{i=0}^{\infty}$ with $\Delta_{i,k}(A) = A$ if $i = k$ and $\Delta_{i,k}(A) = 0$ if $i \neq k$,

where $A \subset R$ is nonempty and compact. Let

$$\mathbf{S}_o = \{E_k(A) : k \in \mathbf{V}, A \subset R \text{ is nonempty and compact}\}.$$

It is easy to see that $\mathcal{T}_v(F) \xrightarrow{d,\varphi,\mathcal{V}} F$ for every $F \in \mathbf{S}_o$. Let $\mathbf{S}$ be the set of all finite linear combinations of elements of the set $\mathbf{S}_o$. From Lemma 3 we easily obtain that $\mathbf{S}_{X_\varphi,d,\varphi,\mathcal{V}} = X_\varphi$. So we easily obtain the assertion from Theorem 4. $\square$

Now, let us denote: $\overline{x}_{j,K_v} = \{\underbrace{0, \ldots, 0}_{j-times}, K_{v,0}, K_{v,1}, \ldots\}$.

From Remark 1 we easily obtain the following extension of Theorem 3 from [3]:

Theorem 6. *Let the assumptions of Lemmas* 1 *and* 3 *hold. If* $\overline{x}_{j,K_v} \xrightarrow{d,\varphi,\mathcal{V}} \mathbf{0}$ *for every* $j \in \mathbf{V}$, *then* $\mathcal{T}_v(F) \xrightarrow{d,\varphi,\mathcal{V}} \mathbf{0}$ *for every* $F \in X_\varphi$.

Let now

$$\overline{X}_{d,\varphi} = \{F \in X_\varphi : F_n \xrightarrow{d,\varphi,\mathcal{V}} F \ \text{ for some } \ F_n \in \tilde{X}'_\varphi, n \in \mathbf{N}\}.$$

Remark 4. For every nonempty and compact $A \in R$ and every $\varepsilon > 0$ there are $n \in \mathbf{N}$ and $a_j \in R,\ j = 0, 1, \dots, n$ such that $\operatorname{dist}(A, \bigcup_{j=0}^{n}\{a_j\}) < \varepsilon$.

From Lemma 3 and Remark 4 we easily obtain the following:

Theorem 7. *If the assumptions of Lemma* 3 *hold, then* $\overline{X}_{d,\varphi} = X_\varphi$.

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