Minimax control of nonlinear evolution equations

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Abstract. In this paper we study the minimax control of systems governed by a nonlinear evolution inclusion of the subdifferential type. Using some continuity and lower semicontinuity results for the solution map and the cost functional respectively, we are able to establish the existence of an optimal control. The abstract results are then applied to obstacle problems, semilinear systems with weakly varying coefficients (e.g. oscillating coefficients) and differential variational inequalities.

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Classification: 49J35, 49J20

1. Introduction

In this paper, we examine a minimax control problem for nonlinear infinite dimensional control systems monitored by evolution equations of the subdifferential type.

So let $T = [0, b]$ and H a separable Hilbert space. We consider a nonlinear control system with dynamics described by

$$(1) \qquad \left\{ \begin{array}{c} -\dot{x}(t) \in \partial\varphi(t, x(t), \lambda) + f_1(t, x(t), \lambda) + f_2(t, x(t), \lambda)u(t) \text{ a.e.} \\ x(0) = x_0(\lambda) \\ u(t) \in U(t) \text{ a.e., } u(\cdot)\text{-measurable.} \end{array} \right\}$$

Here E is a complete metric space and $\lambda \in E$ models noise, disturbance and inaccuracy of measurement, which interfere with the control of the system. Let Y be a separable Banach space, modelling the control space. A control function $u : T \to Y$ is admissible, if $u(\cdot)$ is measurable and $u(t) \in U(t)$ a.e. on T. We will denote the set of admissible controls by S_U. Given $[\lambda, u] \in E \times S_U$, under appropriate hypotheses on the data (see Section 3), we can guarantee the existence of a trajectory (strong solution) $x(\lambda, u)(\cdot) \in C(T, H)$ of (1). Then the performance of the system is evaluated using the cost functional

$$J(\lambda, u) = \int_0^b L(t, x(\lambda, u)(t), \lambda, u(t))\, dt.$$

Since the disturbance $\lambda \in E$ is not a priori known, the system analyst takes a pessimistic approach and tries to minimize the maximum cost. So our optimization problem is the following minimax problem

$$(P) \qquad \sigma = \inf_{u \in S_U} \sup_{\lambda \in E} J(\lambda, u).$$

In what follows for a fixed admissible control $u \in S_U$, we set

$$m(u) = \sup_{\lambda \in E} J(\lambda, u),$$

i.e. $m(u)$ represents the maximum risk associated with the control $u \in S_U$. Hence our goal is to find an admissible control $\widehat{u} \in S_U$ such that

$$\sigma = m(\widehat{u}).$$

Such a control will be called "optimal".

Similar problems for different classes of systems were recently considered by Ahmed [17], [18] and Tanimoto [19]. The reader can consult them and the references therein for further details and different aspects of the problem.

Finally we should mention with some additional effort we can assume that the control constraint set depends on $\lambda \in E$ (i.e. the control is adaptive). However, to keep technical complications to a reasonable level and make the presentation easier to digest, we stay with an open-loop control constraint set.

2. Preliminaries

Let (Ω, Σ) be a measurable space and X a separable Banach space. Throughout this paper we will be using the following notations:

$$P_{f(c)}(X) = \{A \subseteq X : \text{ nonempty, closed, (convex)}\}$$
$$\text{and } \; P_{(w)k(c)}(X) = \{A \subseteq X : \text{ nonempty, (weakly-) compact, (convex)}\}.$$

A multifunction (set-valued function) $F : \Omega \to P_f(X)$ is said to be measurable if for all $x \in X$, $\omega \to d(x, F(\omega)) = \inf\{\|x - z\| : z \in F(\omega)\}$ is measurable. Such a multifunction admits measurable selectors; i.e. there exists $f : \Omega \to X$ measurable, such that $f(\omega) \in F(\omega)$ for all $\omega \in \Omega$ (see Wagner [13, Theorem 4.2]).

Let $\varphi : H \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. We say that $\varphi(\cdot)$ is proper, if it is not identically $+\infty$. Assume that $\varphi(\cdot)$ is proper, convex and lower semicontinuous. We denote the family of such $\overline{\mathbb{R}}$-valued functions by $\Gamma_0(H)$. By $\operatorname{dom} \varphi$, we denote the effective domain of $\varphi(\cdot)$; i.e. $\operatorname{dom} \varphi = \{x \in H : \varphi(x) < \infty\}$. The subdifferential of $\varphi(\cdot)$ at x is the set $\partial\varphi(x) = \{x^* \in H : (x^*, y - x) \le \varphi(y) - \varphi(x) \text{ for all } y \in \operatorname{dom} \varphi\}$, where $(\cdot, \cdot)$ denotes the inner product in the Hilbert space H. If $\varphi(\cdot)$ is Gateaux differentiable at x, then $\partial\varphi(x) = \{\varphi'(x)\}$. We say that $\varphi(\cdot)$ is of compact type, if for every $\theta \in \mathbb{R}_+$, the level set $\{x \in H : \|x\|^2 + \varphi(x) \le \theta\}$ is compact. Also for $\mu > 0$, we set $J_\mu = (I + \mu\partial\varphi)^{-1}$ (the resolvent of $\partial\varphi(\cdot)$). It is well known (see for example Brézis [4]), that for all $\mu > 0$, $D(J_\mu) = H$ and furthermore that $J_\mu(\cdot)$ is nonexpansive.

Let Z be a Banach space and $\{A_n, A\}_{n\ge1} \subseteq 2^Z \setminus \{\emptyset\}$. Let s- denote the strong topology on Z and w- the weak topology on Z. We define

$$\begin{aligned} s\text{-}\underline{\lim} A_n &= \{z \in Z : \lim d(z, A_n) = 0\} \\ &= \{z \in Z : z = s\text{-}\lim z_n,\ z_n \in A_n,\ n \ge 1\} \end{aligned}$$

and $w\text{-}\overline{\lim} A_n = \{z \in Z;\ z = w\text{-}\lim z_{n_k},\ z_{n_k} \in A_{n_k},\ n_1 < n_2 < \cdots < n_k < \dots\}$.

It is clear that we always have $s\text{-}\underline{\lim} A_n \subseteq w\text{-}\overline{\lim} A_n$. If $s\text{-}\underline{\lim} A_n = w\text{-}\overline{\lim} A_n = A$, then we say that the A_n's converge to A in the Mosco sense and denote it by $A_n \xrightarrow{M} A$. Using this concept of set-convergence, we can introduce a new notion of convergence of functions, in general distinct from the pointwise convergence. Recall that if $\varphi : Z \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a proper function, then the epigraph of $\varphi(\cdot)$ is the set $\operatorname{epi}\varphi = \{[z, \theta] \in Z \times \mathbb{R} : \varphi(z) \le \theta\}$. Let $\varphi_n, \varphi : Z \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ be proper functions. We say that the φ_n's converge to φ in the Mosco sense, denoted by $\varphi_n \xrightarrow{M} \varphi$, if and only if $\operatorname{epi}\varphi_n \xrightarrow{M} \operatorname{epi}\varphi$ in $Z \times \mathbb{R}$. For further details on these concepts we refer to Mosco [11] and Attouch [1].

3. Existence of optimal controls

In this section we establish the existence of optimal controls for the minimax problem described in Section 1. Recall H is a separable Hilbert space, Y is a separable reflexive Banach space and E a complete metric space. We will need the following hypotheses on the data.

$\underline{H(\varphi)}$: $\varphi : T \times H \times E \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a function such that

(1) for every $t \in T$ and $\lambda \in E$, $\varphi(t, \cdot, \lambda)$ is proper, convex, l.s.c. (i.e. $\varphi(t, \cdot, \lambda) \in \Gamma_0(H)$),

(2) for any positive integer r, there exists a constant $K_r > 0$, an absolutely continuous function $g_r : T \to \mathbb{R}$ with $\dot{g}_r \in L^\beta(T)$ and a function of bounded variation $h_r : T \to \mathbb{R}$ such that, if $t \in T$, $x \in \operatorname{dom}\varphi(t, \cdot, \lambda)$ with $\|x\| \le r$ and $s \in [t, b]$, $\lambda \in E$, then there exists $\widehat{x} \in \operatorname{dom}\varphi(s, \cdot, \lambda)$ satisfying

$$\begin{aligned} &\|\widehat{x} - x\| \le |g_r(s) - g_r(t)|(|\varphi(t, x, \lambda)| + K_r)^\alpha \\ \text{and}\ \ &\varphi(s, \widehat{x}, \lambda) \le \varphi(t, x, \lambda) + |h_r(s) - h_r(t)|(|\varphi(t, x, \lambda)| + K_r) \end{aligned}$$

where $\alpha \in [0, 1]$ and $\beta = 2$ if $\alpha \in [0, 1/2]$ or $\beta = 1/(1 - \alpha)$ if $\alpha \in [1/2, 1]$,

(3) for every $(t, \lambda) \in (T \setminus N) \times E$, $\mu(N) = 0$ ($\mu(\cdot)$ being the Lebesgue measure on T), $B \subseteq E$ compact and $\theta \in \mathbb{R}_+$, $\overline{\bigcup_{\lambda\in B}\{x \in H : \|x\|^2 + \varphi(t, x, \lambda) \le \theta\}} \in P_{kc}(H)$,

(4) if $\lambda_n \to \lambda \in E$, then $\varphi(t, \cdot, \lambda)$ for all $t \in T \setminus N$.

Remark. Hypotheses $H(\varphi)$ (1) and (2) are essentially due to Kenmochi [9] and Yamada [14]. Here we employ the slightly more general version first introduced by Yotsutani [15]. More precisely in Kenmochi [9], $N = \emptyset$, $\alpha = 0$ and g_r, h_r are both Lipschitz continuous, while in Yamada [14], $N = \emptyset$ and h_r is absolutely continuous.

$\underline{H(f)}$: $f_1 : T \times H \times E \to H$ and $f_2 : T \times H \times E \to \mathcal{L}(Y, H)$ are functions such that

(1) $t \to f_1(t, x, \lambda)$ and $t \to f_2(t, x, \lambda)u$ are measurable for every $(x, \lambda, u) \in H \times E \times Y$,

(2) $\|f_1(t, x, \lambda) - f_1(t, y, \lambda)\| \leq k_B(t)\|x - y\|$ a.e. and $\|f_2(t, x, \lambda) - f_2(t, y, \lambda)\|_{\mathcal{L}} \leq k_B(t)\|x - y\|$ a.e. for all $\lambda \in B \subseteq E$ compact with $k_B \in L^1(T)$,

(3) $\|f_1(t, x, \lambda)\|$, $\|f_2(t, x, \lambda)\|_{\mathcal{L}} \leq a_B(t) + c_B(t)\|x\|$ a.e. for all $\lambda \in B \subseteq E$ compact with $a_B, c_B \in L^2(T)$,

(4) $\lambda \to f_1(t, x, \lambda)$ and $\lambda \to f_2(t, x, \lambda)u$, $f_2(t, x, \lambda)^* h$ are all continuous for every $(t, x, u, h) \in T \times H \times Y \times H$.

$\underline{H(U)}$: $U : T \to P_{wkc}(Y)$ is a measurable multifunction such that $|U(t)| = \sup\{\|u\|_Y : u \in U(t)\} \leq M$.

$\underline{H_0}$: $x_0 : E \to H$ is a continuous map, $x_0(\lambda) \in \operatorname{dom} \varphi(0, \cdot, \lambda)$ and $\sup_{\lambda \in B} \varphi(0, x_0(\lambda), \lambda) < \infty$ for every $B \subseteq E$ compact.

$\underline{H(L)}$: $L : T \times H \times E \times Y \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is an integrand such that

(1) $(t, x, \lambda, u) \to L(t, x, \lambda, u)$ is measurable,

(2) $(x, \lambda, u) \to L(t, x, \lambda, u)$ is l.s.c.,

(3) $u \to L(t, x, \lambda, u)$ is convex,

(4) $\widehat{c}_B(t) - \widehat{a}_B(\|x\| + \|u\|_Y) \leq L(t, x, \lambda, u)$ a.e. with $\widehat{c}_B \in L^1(T)$, $\widehat{a}_B \geq 0$.

$\underline{H_1}$: There exists at least one $u \in S_U$ such that $J(\lambda, u) \leq M_1$ for all $\lambda \in E$ with $M_1 > 0$.

Remark. This last hypothesis guarantees that the value σ of our minimax control problem is in fact finite.

Note that the hypotheses $H(\varphi)$ (1) $\to$ (3), $H(f)$, $H(U)$ and H_0 guarantee that for every pair $[\lambda, u] \in E \times S_U$, there is a strong solution $x(\lambda, u)(\cdot) \in C(T, H)$ of (1) (cf. Theorem 4.1 of Papageorgiou [12]). Furthermore because of the hypothesis $H(f)$ (2) and due to the monotonicity of the subdifferential operator, we readily see that $x(\lambda, u)(\cdot) \in C(T, H)$ is unique.

The following technical lemma due to Yotsutani [15] (cf. Lemmata 3.4 and 4.4) will be used in the sequel. For easy reference, we state it here without a proof, which can be found in [15].

Lemma 3.1. *<u>If</u> hypotheses $H(f)$ (1) and (2) hold, <u>then</u>*

(a) *for each $x : T \to H$ measurable, $t \to \varphi(t, x(t), \lambda)$ is measurable,*

(b) *if $\Phi : L^2(T, H) \times E \to \overline{\mathbb{R}}$ is defined by*

$$\Phi(z, \lambda) = \begin{cases} \int_0^b \varphi(t, z(t), \lambda)\, dt & \text{if } \varphi(\cdot, z(\cdot), \lambda) \in L^1 \\ +\infty & \text{otherwise} \end{cases}$$

then $h \in \partial\Phi(z,\lambda)$ *if and only if* $h(t) \in \partial\varphi(t, z(t), \lambda)$ *a.e. on* T.

Our first result establishes a continuity property of the map $(\lambda, u) \to x(\lambda, u)(\cdot)$, from $E \times S_U$ into $C(T, H)$. In what follows the set $S_U \subseteq L^2(T, Y)$ is furnished with the relative weak $L^2(T, Y)$-topology. Then S_U topologized like that, is compact and metrizable (cf. hypothesis $H(U)$).

Proposition 3.2. *If hypotheses $H(\varphi)$, $H(f)$, $H(U)$, and H_0 hold, then $x : E \times S_U \to C(T, H)$ is continuous.*

Proof: Let $[\lambda_n, u_n] \to [\lambda, u]$ in $E \times S_U$. For every $n \geq 1$, let $y_n(\cdot) \in C(T, H)$ be the unique strong solution of

$$-\dot{y}_n(t) \in \partial\varphi(t, y_n(t), \lambda_n) \text{ a.e.}, \;\; y_n(0) = x_0(\lambda_n).$$

Exploiting the monotonicity of the subdifferential and using Lemma A.5, p. 157 of Brézis [4], we get that

$$\|x_n(t) - y_n(t)\| \leq \int_0^t \|f_1(s, x_n(s), \lambda_n) + f_2(s, x_n(s), \lambda_n)u_n(s)\| \, ds, \;\; t \in T$$

$$\Rightarrow \|x_n(t)\| \leq \sup_{n \geq 1} \|y_n\|_\infty + \int_0^t (1 + M)(a_B(s) + c_B(s)\|x_n(s)\| \, ds)$$

where $B = \{\lambda_n, \lambda\}_{n \geq 1}$ (cf. the hypothesis $H(f)$). From Theorem 3.1 of [12] we know that $\sup_{n\geq 1} \|y_n\|_\infty < \infty$. Hence using Gronwall's inequality, we get $M_{1B} > 0$ such that $\sup_{n \geq 1} \|x_n(t)\| \leq M_{1B}$ for all $t \in T$. So without any loss of generality we may assume that $\|f_1(t, x, \lambda_n)\|$, $\|f_2(t, x, \lambda_n)\|_{\mathcal{L}} \leq a_B(t) + c_B(t)M_{1B} = \psi_B(t)$ a.e. for all $(t, x) \in T \times H$ and all $n \geq 1$. Let $\Gamma_B = \{h \in L^2(T, H) : \|h(t)\| \leq \psi_B(t) \text{ a.e.}\}$ and $K_B = p(\Gamma_B, B)$, where $p : L^2(T, H) \times E \to C(T, H)$ is the map which to each $h \in \Gamma_B$, $\lambda \in B$ and each $z_0 \in x_0(B) \in P_k(H)$ (cf. hypothesis H_0), assigns the unique solution of $-\dot{x}(t) \in \partial\varphi(t, x(t), \lambda) + h(t)$ a.e., $x(0) = z_0$. Then for every $x(\cdot) \in K_B$ we have $\|x(t') - x(t)\| \leq \int_0^t \|\dot{x}(s)\| \, ds \leq (\int_0^b \chi_{[t,t']}(s)^2 \, ds)^{1/2} (\int_0^b \|\dot{x}(s)\|^2 \, ds)^{1/2}$ and from inequality (7.5) of Yotsutani [15] we get that there exists $M_{2B} > 0$ such that $\|\dot{x}\|_{L^2(T,H)} \leq M_{2B}$ for all $x(\cdot) \in K_B$. So $\|x(t') - x(t)\| \leq (t' - t)^{1/2} M_{2B}$, for all $x(\cdot) \in K_B$. Therefore K_B is equicontinuous. Also because of hypothesis $H(\varphi)\,(3)$ $K_B(t) = \{x(t) : x(\cdot) \in K_B\}$ is relatively compact. So by the Arzela-Ascoli theorem, $K_B \subseteq C(T, H)$ is relatively compact and so we may assume $x_n \to x$ in $C(T, H)$ while clearly we can also say $\dot{x}_n \xrightarrow{w} y$ in $L^2(T, H)$. Evidently $x(t) = x_0(\lambda) + \int_0^t y(s) \, ds$, $t \in T$; i.e. $y = \dot{x}$. We will show that $x = x(\lambda, u)$. Let $h \in L^2(T, H)$. We have

$$\begin{aligned}
&(h, f_2(\cdot, x_n(\cdot), \lambda_n)u_n(\cdot))_{L^2(T,H)} \\
&= \int_0^b (h(t), f_2(t, x_n(t), \lambda_n)u_n(t)) \, dt \\
&= \int_0^b (f_2(t, x_n(t), \lambda_n)^* h(t), u_n(t)) \, dt.
\end{aligned}$$

Because of the hypotheses $H(f)$ (2) and (4), we have that

$$f_2(t, x_n(t), \lambda_n)^* h(t) \longrightarrow f_2(t, x(t), \lambda)^* h(t) \text{ a.e.}$$

and so from the dominated convergence theorem (cf. the hypothesis $H(f)$ (3)), we get that

$$f_2(\cdot, x_n(\cdot), \lambda_n)^* h(\cdot) \longrightarrow f_2(\cdot, x(\cdot), \lambda)^* h(\cdot) \text{ in } L^1(T, Y).$$

Note that because of the hypothesis $H(U)$ and since $u_n \xrightarrow{w} u$ in $L^2(T, Y)$, we also have that $u_n \xrightarrow{w^*} u$ in $L^\infty(T, Y)$. Therefore

$$\begin{aligned}
&\int_0^b (f_2(t, x_n(t), \lambda_n)^* h(t), u_n(t))_{Y^*Y}\, dt \to \int_0^b (f_2(t, x(t), \lambda)^* h(t), u(t))_{Y^*Y}\, dt \\
&\Rightarrow \int_0^b (h(t), f_2(t, x_n(t), \lambda_n) u_n(t))\, dt \to \int_0^b (h(t), f_2(t, x(t), \lambda) u(t))\, dt \\
&\Rightarrow (h, f_2(\cdot, x_n(\cdot), \lambda_n) u_n)_{L^2(T,H)} \to (h, f_2(\cdot, x(\cdot), \lambda) u(\cdot))_{L^2(T,H)}.
\end{aligned}$$

Since $h \in L^2(T, H)$ was arbitrary, we deduce that $f_2(\cdot, x_n(\cdot), \lambda_n) u_n(\cdot) \xrightarrow{w} f_2(\cdot, x(\cdot), \lambda) u(\cdot)$ in $L^2(T, H)$.

Next let $\Phi : L^2(T, H) \times E \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ be defined by

$$\Phi(z, \lambda) = \begin{cases} \int_0^b \varphi(t, z(t), \lambda)\, dt & \text{if } \varphi(\cdot, z(\cdot), \lambda) \in L^1(T) \\ +\infty & \text{otherwise.} \end{cases}$$

Note that Lemma 3.1 above tells us that for every $z \in L^2(T, H)$ the function $t \to \varphi(t, z(t), \lambda)$ is measurable and so the functional $\Phi(\cdot, \lambda)$ is well defined and in fact proper (cf. Corollary 4.1 of Yotsutani [15]). Furthermore using once more Lemma 3.1 above, we know that if $v \in L^2(T, H)$, $\varphi(\cdot, v(\cdot), \lambda) \in L^1(T)$ and $w \in L^2(T, H)$, then we have that $w \in \partial\Phi(v, \lambda)$ if and only if $w(t) \in \partial\varphi(t, x(t), \lambda)$ a.e. Hence we have that

$$[x_n - \dot{x}_n - f_1(\cdot, x_n(\cdot), \lambda_n) - f_2(\cdot, x_n(\cdot), \lambda_n) u_n(\cdot)] \in Gr\partial\Phi(\cdot, \lambda_n)$$

where $Gr\partial\Phi(\cdot, \lambda_n)$ denotes the graph of the subdifferential operator $\partial\Phi(\cdot, \lambda_n)$. We will show that $Gr\partial\Phi(\cdot, \lambda_n) \xrightarrow{K} Gr\partial\Phi(\cdot, \lambda)$ as $n \to \infty$. Here $\xrightarrow{K}$ denotes the Kuratowski convergence of sets (i.e. $s\text{-}\underline{\lim}\, Gr\partial\Phi(\cdot, \lambda_n) = Gr\partial\Phi(\cdot, \lambda) = s\text{-}\overline{\lim}\, Gr\partial\Phi(\cdot, \lambda_n)$) with $s\text{-}\overline{\lim}\, Gr\partial\Phi(\cdot, \lambda_n) = \{[v, w] \in L^2(T, H) : [v, w] = s\text{-}\lim[v_{n_k}, w_{n_k}],\ [v_{n_k}, w_{n_k}] \in Gr\partial\Phi(\cdot, \lambda_{n_k}),\ n_1 < n_2 < \cdots < n_k < \dots\}$, see for example Attouch [1] and Dal Maso [7]). Indeed note that for every $v \in L^2(T, H)$

$$[(I + \partial\Phi(\cdot, \lambda_n))^{-1} v](t) = (I + \partial\varphi(t, \cdot, \lambda_n))^{-1} v(t) \text{ a.e.}$$

and because of hypothesis $H(\varphi)$ (4) and Theorem 3.66, p. 373 of Attouch [1], we have that

$$(I + \partial\varphi(t,\cdot,\lambda_n))^{-1}v(t) \to (I + \partial\varphi(t,\cdot,\lambda))^{-1}v(t) \text{ a.e. in } H.$$

Furthermore from Lemma 3.4 of Yotsutani [15], we know that

$$\|(I + \partial\varphi(t,\cdot,\lambda_n))^{-1}v(t)\| \leq \|v(t)\| + c \text{ a.e.}$$

with $c > 0$ independent of $n \geq 1$. Thus via the dominated convergence theorem, we get

$$(I + \partial\Phi(\cdot,\lambda_n))^{-1}v \longrightarrow (I + \partial\Phi(\cdot,\lambda))^{-1}v \ \text{ in } \ L^2(T,H)$$

and so from Proposition 3.60, p. 361–362 of Attouch [1], we have that

$$\partial\Phi(\cdot,\lambda_n) \xrightarrow{K} \partial\Phi(\cdot,\lambda) \ \text{ as } \ n \to \infty.$$

Note that $f_1(\cdot,x_n(\cdot),\lambda_n) \to f_1(\cdot,x(\cdot),\lambda)$ in $L^2(T,H)$ (cf. hypothesis $H(f)$). So we have $[x_n, -\dot{x}_n - f_1(\cdot,x_n(\cdot),\lambda_n) - f_2(\cdot,x_n(\cdot),\lambda_n)u_n(\cdot)] \to [x, -\dot{x} - f_1(\cdot,x(\cdot),\lambda) - f_2(\cdot,x(\cdot),\lambda)u(\cdot)]$ in $L^2(T,H) \times L^2(T,H)_w$. Therefore, from Proposition 3.59, p. 361 of Attouch [1], we get

$$\begin{gathered}
[x, -\dot{x} - f_1(\cdot,x(\cdot),\lambda) - f_2(\cdot,x(\cdot),\lambda)u(\cdot)] \in Gr\partial\Phi(\cdot,\lambda) \\
\Rightarrow -\dot{x}(t) \in \partial\varphi(t,x(t),\lambda) + f_1(t,x(t),\lambda) + f_2(t,x(t),\lambda)u(t) \text{ a.e.} \\
x(0) = x_0(\lambda) \\
u(t) \in U(t) \text{ a.e., } u(\cdot)\text{-measurable.} \\
\Rightarrow x = x(\lambda,u) \\
\Rightarrow x(\cdot,\cdot) \ \text{ is continuous as claimed.}
\end{gathered}$$

□

The next proposition establishes the lower semicontinuity of the cost functional.

Proposition 3.3. *If hypotheses $H(\varphi)$, $H(f)$, $H(U)$, H_0 and $H(L)$ hold, then $(\lambda,u) \to J(\lambda,u)$ is sequentially l.s.c. from $E \times L^2(T,Y)_w$ into $\overline{\mathbb{R}}$.*

PROOF: We need to show that for every $\theta \in \mathbb{R}$, the level set

$$\Sigma(\theta) = \{[\lambda,u] \in E \times L^2(T,Y) : J(\lambda,u) \leq \theta\}$$

is sequentially closed in $E \times L^2(T,Y)_w$. So let $[\lambda_n,u_n] \in \Sigma(\theta)$, $n \geq 1$, and assume that $[\lambda_n,u_n] \to [\lambda,u]$ in $E \times L^2(T,Y)_w$. Recall that since by hypothesis E is complete, is isometrically isomorphic to a closed subset of a Banach space Z. For $B = \{\lambda_n,\lambda\}_{n\geq 1} \subseteq E$ compact, let Z_B be the Banach subspace of Z generated by the isometric image of B. Clearly Z_B is a separable Banach space (in fact Z_B

is compactly generated). Then because of hypothesis $H(L)$ and Theorem 2.1 of Balder [2], we have that $(x, \lambda, u) \to \int_0^b L(t, x(t), \lambda, u(t))\, dt$ is l.s.c. on $L^1(T, H) \times Z_B \times L^1(T, Y)_w$. Since by Proposition 3.2 $x_n = x(\lambda_n, u_n) \to x = x(\lambda, u)$ in $C(T, H)$, we have

$$\begin{aligned}\int_0^b L(t, x(t), \lambda, u(t))\, dt &\leq \underline{\lim} \int_0^b L(t, x_n(t), \lambda_n, u_n(t))\, dt \\ &\Rightarrow J(\lambda, u) \leq \underline{\lim} J(\lambda_n, u_n) \leq \theta \\ &\Rightarrow [\lambda, u] \in \Sigma(\theta) \\ &\Rightarrow J(\cdot, \cdot) \ \text{ is sequentially l.s.c. as claimed.}\end{aligned}$$

□

Before going into our main existence theorem concerning the minimax control problem under consideration, let us briefly elaborate on hypothesis H_1. As we already mentioned that hypothesis guarantees the finiteness of the value σ of our problem. Hypothesis H_1 is valid, if E is compact and for all $\lambda \in E$ and $x \in H$ with $\|x\| \leq r$, we have $L(t, x, \lambda, u) \leq \psi_r(t)$ a.e. for all $u \in Y$, $\|u\| \leq M$, with $\psi_r(\cdot) \in L^1(T)$. Indeed to see this, fix $u \in S_U$ and let $y(\lambda)(\cdot) \in C(T, H)$ be the unique strong solution of the Cauchy problem

$$\left\{ \begin{array}{c} -\dot{y}(t) \in \partial\varphi(t, y(t), \lambda) \text{ a.e.} \\ y(0) = x_0(\lambda). \end{array} \right\}$$

Let $x = x(\lambda, u)$. Exploiting the monotonicity of the subdifferential, we have

$$\begin{aligned}&(-\dot{x}(t) + y(\lambda)(t), y(\lambda)(t) - x(t)) \\ &\leq (f_1(t, x(t), \lambda) + f_2(t, x(t), \lambda)u(t), y(\lambda)(t) - x(t)) \text{ a.e.} \\ \Rightarrow \frac{1}{2}\frac{d}{dt}\|x(t) - y(\lambda)(t)\|^2 &\leq (f_1(t, x(t), \lambda) + f_2(t, x(t), \lambda)u(t), y(\lambda)(t) - x(t)) \text{ a.e.} \\ &\Rightarrow \|x(t) - y(\lambda)(t)\|^2 \\ &\leq 2\int_0^t \|f_1(s, x(s), \lambda) + f_2(s, x(s), \lambda)u(s)\| \cdot \|y(\lambda)(s) - x(s)\|\, ds.\end{aligned}$$

Invoking Lemma A.5, p. 157 of Brézis [4], we get that

$$\begin{aligned}\|x(t) - y(\lambda)(t)\| &\leq 2\int_0^t \|f_1(s, x(s), \lambda) + f_2(s, x(s), \lambda)u(s) - x(s)\|\, ds \\ &\leq 2\int_0^t (M+1)(a(s) + c(s)\|x(s)\|)\, ds \\ \Rightarrow \|x(t)\| &\leq \sup_{\lambda \in E} \|y(\lambda)(t)\| + 2\int_0^t (M+1)(a(s) + c(s)\|x(s)\|)\, ds.\end{aligned}$$

But from Proposition 3.1, we know that $\sup_{\lambda \in E} \|y(\lambda)(t)\| = \gamma < \infty$ for all $t \in T$ (recall that E is assumed to be compact). So invoking Gronwall's lemma, we get that for all $\lambda \in E$ we have

$$\|x(\cdot)\|_{C(T,H)} \leq r, \quad r > 0.$$

Therefore $m(u) \leq \|\psi_r\|_1 \Rightarrow \sigma$ is finite (cf. the hypotheses $H(U)$ and $H(L)$).

Now we are ready to state our main result concerning problem (P).

Theorem 3.4. *If the hypotheses $H(\varphi)$, $H(f)$, $H(U)$, $H(L)$, H_0 and H_1 hold, then problem (P) admits an optimal control.*

PROOF: From Proposition 3.3, we know that $(\lambda, u) \to J(\lambda, u)$ is sequentially l.s.c. on $E \times L^2(T, Y)_w$. So Theorem 1, p. 122 of Berge [3] tells us that $u \to \sup_{\lambda \in E} J(\lambda, u) = m(u)$ is sequentially l.s.c. on $L^2(T, Y)_w$. Then since S_U is sequentially weakly compact in $L^2(T, Y)$ (being weakly closed and bounded), we get that $\inf\{m(u) : u \in S_U\}$ admits a solution $\widehat{u} \in S_U$. Clearly this is the desired optimal control for our minimax problem (P). $\square$

4. Applications

In this section we work out in detail three examples, illustrating the abstract results of Section 3.

(A) Minimax control of obstacles.

Let $T = [0, b]$ and $Z \subseteq \mathbb{R}^N$ be a bounded domain with smooth boundary $\Gamma = \partial Z$. Let $2 \leq p < \infty$, $g : T \times Z \to \mathbb{R}$ and for every $t \in T$ define

$$K(t) = \{h \in W^{1,p}(Z) : g(t, z) \leq h(z) \text{ a.e. on } Z\}.$$

Clearly this is a closed and convex subset of $W^{1,p}(Z)$ (the obstacle). The dynamics of our distributed parameters systems are described by the following parabolic variational inequality (here $D_k = \frac{\partial}{\partial z_k}$, $k = 1, \ldots, N$, while $D = (D_k)_{k=1}^N$, the gradient operator). Also $N \leq \frac{2p}{p-2}$ if $p \neq 2$.

$$(2) \quad \left\{ \begin{gathered} \frac{\partial x}{\partial t} - \sum_{k=1}^N D_k(a(z, \lambda)|D_k x|^{p-2} D_k x) \geq f(t, z, x(t, z), \lambda) + b(t, z, \lambda) u(t, z) \\ x(t, z) \geq g(t, z) \text{ a.e. on } T \times Z, \\ \Big(\frac{\partial x}{\partial t} - \sum_{k=1}^N D_k(a(z, \lambda), |D_k x|^{p-2} D_k x) \\ -f(t, z, x(t, z), \lambda) - b(t, z, \lambda) u(t, z)\Big)(x(t, z) - g(t, z)) = 0 \\ x(0, z) = x_0(z, \lambda) \text{ a.e. on } Z, \\ \|u(t, \cdot)\|_2 \leq M \text{ a.e., } u(\cdot, \cdot)\text{-measurable.} \end{gathered} \right\}$$

Our cost functional is given by

$$J(\lambda, u) = \int_0^b \int_Z L(t, z, x(\lambda, u)(t, z), \lambda, u(t, z))\, dz\, dt$$

and our task is to solve the minimax problem

$$(P)_1 \qquad \inf \left[\sup_{\lambda \in E} J(\lambda, u) : \|u(t, \cdot)\|_2 \le M \right] = \sigma,$$

i.e. find a control $\widehat{u} \in L^2(T \times Z)$ such that $\|\widehat{u}(t, \cdot)\|_2 \le M$ and $m(\widehat{u}) = \sigma$, where as before $m(u) = \sup_{\lambda \in E} J(\lambda, u)$.

We will need the following hypotheses on the data:

$\underline{H(g)}$: $g : T \times Z \to \mathbb{R}$ is a function such that

(1) for every $t \in T$, $g(t, \cdot) \in W^{1,\infty}(Z)$,
(2) there exist $0 < c_1 \le c_2$ such that $c_1 \le g(t, z) \le c_2$ and $\|Dg(t, z)\|_N \le c_2$ for all $t \in T$ and almost all $z \in Z$ (here $\|\cdot\|_N$ denotes the ℓ^p-norm of $\mathbb{R}^N$),
(3) there exist $r_1 \in W^{1,2}(T)$ and $r_2 : T \to \mathbb{R}$ a functional of bounded variation such that for all $t \in T \setminus N$, $\mu(N) = 0$ (as before $\mu(\cdot)$ is the Lebesgue measure on T), we have

$$|g(t, z) - g(s, z)| \le |r_1(t) - r_1(s)| \text{ a.e. on } Z$$

and $\|Dg(t, \cdot) - Dg(s, \cdot)\|_{L^2(Z, \mathbb{R}^N)} \le |r_2(t) - r_2(s)|$.

$\underline{H(a)}$: $0 < m_{1B} \le a(z, \lambda) \le m_{2B}$ a.e. on Z for all $\lambda \in B \subseteq E$ compact and if $\lambda_n \Rightarrow \lambda$ in E, then $a(z, \lambda_n) \to a(z, \lambda)$ a.e. on Z.

$\underline{H(f)_1}$: $f : T \times Z \times \mathbb{R} \times E \to \mathbb{R}$ is a function such that

(1) $(t, z) \to f(t, z, x, \lambda)$ is measurable,
(2) $|f(t, z, x, \lambda) - f(t, z, y, \lambda)| \le k_B(t, z)|x - y|$ a.e. for all $\lambda \in B \subseteq E$ compact with $k_B \in L^1(T \times Z)$,
(3) $|f(t, z, x, \lambda)| \le a_B(t, z) + c_B(t, z)|x|$ a.e. for all $\lambda \in B \subseteq E$ compact and with $a_B \in L^2(T \times Z)$, $c_B \in L^2(T, L^\infty(Z))$,
(4) $\lambda \to f(t, z, x, \lambda)$ is continuous.

$\underline{H(b)}$: $b(\cdot, \cdot, \lambda) \in L^\infty(T \times Z)$, $\lambda \to b(t, z, \lambda)$ is continuous and $|b(t, z, \lambda)| \le \eta(t, z)$ a.e. $\eta \in L^2(T \times Z)$, $\lambda \in B \subseteq E$ compact.

$\underline{H(L)_1}$: $L : T \times Z \times \mathbb{R} \times E \times \mathbb{R} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is an integrand such that

(1) $(t, z, x, \lambda, u) \to L(t, z, x, \lambda, u)$ is measurable,
(2) $(x, \lambda, u) \to L(t, z, x, \lambda, u)$ is l.s.c.,
(3) $u \to L(t, z, x, \lambda, u)$ is convex,
(4) $c_B^1(t, z) - a_B^1(z)(|x| + |u|) \le L(t, z, x, \lambda, u)$ a.e. for all $\lambda \in B \subseteq E$ compact and with $c_B^1 \in L^1(T \times Z)$, $a_B^1 \in L^1(Z)$.

$\underline{H_0'}$: $\lambda \to x_0(\cdot,\lambda)$ is continuous from E into $L^2(Z)$, $x_0(\cdot,\lambda) \in K(0)$ and $\sup_{\lambda\in B} \|Dx_0(\cdot,\lambda)\|_{L^p(Z,\mathbb{R}^N)} < \infty$.

$\underline{H_1'}$: there exists $u \in L^2(T \times Z)$, $\|u(t,\cdot)\|_2 \le M$ a.e. such that for all $\lambda \in E$

$$\int_0^b \int_Z L(t,z,x(\lambda,u)(t,z),\lambda,u(t,z))\,dz\,dt \le \gamma < \infty.$$

We have the following existence result concerning problem $(P)_1$.

Theorem 4.1. *$\underline{\text{If}}$ the hypotheses $H(g)$, $H(a)$, $H(f)_1$, $H(b)$, $H(L)_1$, H_0' and H_1' hold, $\underline{\text{then}}$ problem $(P)_1$ admits an optimal control.*

Proof: Let $H = L^2(Z)$ and define $\varphi : T \times H \times E \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ by

$$\varphi(t,x,\lambda) = \begin{cases} \frac{1}{p}\int_Z a(z,\lambda)\|Dx(z)\|_N^p\,dz & \text{if } x \in K(t) \\ +\infty & \text{otherwise.} \end{cases}$$

It is easy to check that for all $(t,\lambda) \in T \times E$, $\varphi(t,\cdot,\lambda) \in \Gamma_0(H)$. Next let $s,t \in T$ and $x(\cdot) \in K(t)$. Define $y(z) = \frac{g(s,z)}{g(t,z)}x(z)$. Clearly $y(\cdot) \in K(s)$ and we can easily check that

$$\int_Z |y(z)-x(z)|^2\,dz \le \frac{1}{c_1^2}|r_1(s)-r_1(t)|^2 \int_Z |x(z)|^2\,dz.$$

So if $\|x\|_2 \le r$, then $\|x-y\|_{L^2(Z)} \le c_3|r_1(t)-r_1(s)|^2 r$ with $c_3 = \frac{1}{c_1^2}$ and so we have satisfied the first inequality in the hypothesis $H(\varphi)$ (2), with $g_r(t) = c_3 r_1(t) r$. Also we have

$$\begin{gathered} \|Dy(z)\|_N^p - \|Dx(z)\|_N^p \\ \le c_4\left(\left|\frac{g(s,z)}{g(t,z)}\right|^p - 1\right)\|Dx(z)\|_N^p + c_5\frac{\|g(t,z)Dg(s,z)-g(s,z)Dg(t,z)\|_N^p}{c_1^p} \end{gathered}$$

for some $c_4 > 0$ (independent of (t,s,x)), large enough (i.e. $c_4 > 2^{p-1}$ and $\left(\frac{c_2}{c_1}\right)^p \le \frac{c-2^{p-1}}{c-1}$). Then by elementary algebraic calculations, we finally get $c_5 > 0$ (independent of (t,s,x)) such that

$$|\varphi(t,x,\lambda)-\varphi(s,y,\lambda)| \le c_6(|r_1(t)-r_1(s)|+|r_2(t)-r_2(s)|)(1+\varphi(t,x,\lambda)).$$

So we have satisfied the hypotheses $H(\varphi)$ (1) and (2), with $\alpha = \frac{1}{p} \in [0,1/2]$, hence $\beta = 2$, and with $g_r = c_3 r_1(t) r$ and $h_r(\cdot) = c_5(r_1(\cdot)+r_2(\cdot))$.

Let $B \subseteq E$ be compact. Then because of the hypothesis $H(a)$, we can easily see that $\bigcup_{\lambda\in B}\{x \in H = L^2(Z) : \|x\|^2 + \varphi(t,x,\lambda) \le \theta\}$ is bounded in $W^{1,p}(Z)$, hence relatively sequentially weakly compact. Since $W^{1,p}(Z)$ embeds compactly in $L^2(Z)$ (recall $p \geq 2$), we conclude that the hypothesis $H(\varphi)$ (3) is satisfied.

Finally we will show that if $x_n \xrightarrow{w} x$ in $L^2(Z)$ and $\lambda_n \to \lambda$ in E, then $\varphi(t,x,\lambda_n) \leq \underline{\lim}\varphi(t,x_n,\lambda_n)$. Assume that $\underline{\lim}\varphi(t,x_n,\lambda_n) < +\infty$ (or otherwise the inequality is automatically true). Let $\{x_{n_k}\}_{k\geq 1}$ be a subsequence such that $\lim\varphi(t,x_{n_k},\lambda_{n_k}) = \underline{\lim}\varphi(t,x_n,\lambda_n)$. Then from the definition of $\varphi(t,x,\lambda)$ we have that $\{x_{n_k}\}_{k\geq 1} \subseteq K(t)$ is bounded, hence relatively sequentially weakly compact in $W^{1,p}(Z)$. Thus we assume that $x_{n_k} \xrightarrow{w} x$ in $W^{1,p}(Z)$. But then from the hypothesis $H(a)$ and Theorem 5.14, p. 51 of Dal Maso [7], we have that

$$\int_Z a(z,\lambda)\|Dx(z)\|_N^p\,dz \leq \underline{\lim}\int_Z a(z,\lambda_{n_k})\|Dx_{n_k}(z)\|^p\,dz$$
$$\Rightarrow \varphi(t,x,\lambda) \leq \underline{\lim}\varphi(t,x_n,\lambda_n).$$

On the other hand, because of the hypothesis $H(a)$, $\varphi(t,x,\lambda_n) \to \varphi(t,x,\lambda)$. So Lemma 1.10 of Mosco [11] tells us that $\varphi(t,\cdot,\lambda_n) \xrightarrow{M} \varphi(t,\cdot,\lambda)$ in $H = L^2(Z)$.

Next let $f_1 : T\times H\times E \to H$ and $\widehat{b} : T\times E \to H$ be the Nemitsky (superposition) operators corresponding to f and b respectively; i.e. $f_1(t,x,\lambda)(z) = f(t,z,x(z),\lambda)$, $\widehat{b}(t,\lambda)(z) = b(t,z,\lambda)$. Then using the hypotheses $H(f)_1$ and $H(b)$ we can see that $H(f)$ is valid. Also $U(t) = B_M(0) = \{u \in Y = L^2(Z) : \|u\|_2 \leq M\}$ and $\widehat{L}(t,x,\lambda,u) = \int_Z L(t,z,x(z),\lambda,u(z))\,dz$. Because of the hypothesis $H(L)_1$ and Theorem 2.1 of Balder [2] we see that $\widehat{L}$ satisfies the hypothesis $H(L)$.

Now let $x^* \in \partial\varphi(t,x,\lambda) \subseteq L^2(Z)$. Immediately we can check that $x \in K(t)$. Also if $e \in W^{1,p}(Z)_+$ (the positive cone of $W^{1,p}(Z)$), we see that

$$\lim_{\delta\downarrow 0}\frac{\varphi(t,x+\delta e,\lambda)-\varphi(t,x,\lambda)}{\delta} = \varphi'(t,x,\lambda;e)$$

where $\varphi'(t,x,\lambda;e)$ is the directional derivative of $\varphi(t,\cdot,\lambda)$ in the direction e. Since $\varphi'(t,x,\lambda;e) = \int_Z\sum_{k=1}^N a(z,\lambda)|D_kx|^{p-2}D_kxD_ke\,dz$, from the definition of the subdifferential we have that $\int_Z x^*(z)e(z)\,dz \leq \int_Z\sum_{k=1}^N a(z,\lambda)|D_kx|^{p-2}D_kxD_ke\,dz$ for every $e \in W^{1,p}(Z)_+$. In addition, if $\delta \in [0,1]$, then $x+\delta(g(t,\cdot)-x) \in K(t)$. So as above

$$\int_Z x^*(z)(x(z)-g(t,z))\,dz \geq \int_Z\sum_{k=1}^N a(z,\lambda)|D_kx|^{p-2}D_kxD_k(x(z)-g(t,z))\,dz.$$

Since $x-g(t,\cdot) \in W^{1,p}(Z)_+$, we deduce that

$$\int_Z x^*(z)(x(z)-g(t,z))\,dz = \int_Z\sum_{k=1}^N a(z,\lambda)|D_kx|^{p-2}D_kxD_k(x(z)-g(t,z))\,dz.$$

Therefore (2) is equivalent to the following subdifferential evolution equation:

$$(2)' \qquad \left\{\begin{array}{c} -\dot{x}(t) \in \partial\varphi(t,x(t),\lambda) + f_1(t,x(t),\lambda) + \widehat{b}(t,\lambda)u(t) \text{ a.e.} \\ x(0) = \widehat{x}_0(\lambda) \quad (\widehat{x}_0(\lambda)(\cdot) = x_0(\cdot,\lambda)) \\ u(t) \in B_M(0) \text{ a.e., } u(\cdot)\text{-measurable.} \end{array}\right\}$$

Let $\widehat{J}(\lambda,u) = \int_0^b \widehat{L}(t,x(\lambda,u)(t),\lambda,u(t))\,dt$. Then $(P)_1$ is equivalent to

$$(P)_1' \qquad \inf\left[\sup_{\lambda\in E} \widehat{J}(\lambda,u) : \|u(t)\|_2 \le M\right] = \sigma$$

and because of the hypotheses H_0' and H_1', we can apply Theorem 3.3 and get that $(P)_1'$ (hence equivalently $(P)_1$) admits an optimal control. $\square$

(B) Minimax control of semilinear systems.

Let $T = [0,b]$, $Z = (0,1)$ and assume that E is a compact metric space. We consider the following semilinear distributed parameters system:

(3)

$$\left\{\begin{array}{c} \dfrac{\partial x}{\partial t} - \dfrac{\partial}{\partial z}(a(t,z,\lambda)\dfrac{\partial x}{\partial z}) + \beta(x(t,z)) = f(t,z,x(t),\lambda) + b(t,z,\lambda)u(t,z) \text{ a.e.} \\ x(t,0) = x(t,1) = 0,\ x(0,z) = x_0(z,\lambda) \text{ a.e. on } Z, \\ \|u(t,\cdot)\|_2 \le M \text{ a.e., } u(\cdot,\cdot)\text{-measurable.} \end{array}\right\}$$

Our cost criterion is the following least squares (quadratic) criterion

$$\inf\left\{\sup_{\lambda\in E} J(\lambda,u) : \|u(t,\cdot)\|_2 \le M\right\} = \sigma.$$

We will need the following hypotheses on the data:

$\underline{H(a)_1}$: $0 < c_1 \le a(t,z,\lambda) \le c_2$ for all $\lambda \in E$ and almost all $(t,z) \times T \times Z$, $|a(t,z,\lambda) - a(s,z,\lambda) \le \widehat{k}|t-s|$ a.e. on Z for all $\lambda \times E$ with $\widehat{k} > 0$ and if $\lambda_n \to \lambda$, then $\frac{1}{a(t,\cdot,\lambda_n)} \xrightarrow{w^*} \frac{1}{a(t,\cdot,\lambda)}$ a.e. in $L^\infty(0,1)$. Also $a(t,\cdot,\lambda) \in C^1(\overline{Z})$.

$\underline{H(\beta)}$: β is a maximal monotone set in $\mathbb{R}\times\mathbb{R}$. Hence $\beta = \partial j$ with $j(\cdot) \in \Gamma_0(\mathbb{R})$ (cf. Brézis [4]). We will assume that $j \ge 0$ and $j(\cdot)$ is continuous (e.g. $j(x) = \frac{kx^2}{2}$, $j(x) = |x|$, $j(x) = e^x$, etc.).

$\underline{H(v)}$: for every $\lambda \in E$, $v(\cdot,\cdot,\lambda) \in L^2(T\times Z)$, $\lambda \to v(\cdot,\cdot,\lambda)$ is sequentially weakly continuous from E into $L^2(T\times Z)$ and $|v(t,z,\lambda)| \le \gamma(t,z)$ a.e. with $\gamma(\cdot,\cdot) \in L^2(T\times Z)$.

Remark. Note that this hypothesis, together with the assumed compactness of E, implies that σ is finite.

$\underline{H_0''}$: $\lambda \to x_0(\cdot,\lambda)$ is continuous from E into $L^2(Z)$, $x_0(\cdot,\lambda) \in H_0^1(Z)$ and $\sup_{\lambda\in E}\int_0^1 j(x_0(z,\lambda))\,dz < \infty$.

Theorem 4.2. *If the hypotheses $H(a)_1$, $H(\beta)$, $H(f)$, $H(b)$, $H(v)$ and H_0'' hold, then the problem $(P)_2$ admits an optimal control.*

Proof: Let $H = L^2(0,1)$ and define $\varphi : T \times H \times E \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ by

$$\varphi(t,x,\lambda) = \begin{cases} \frac{1}{2}\int_0^1 a(t,z,\lambda)\frac{dx}{dz}\,dz + \int_0^1 j(x(z))\,dz & \text{if } x \in H_0^1(0,1) \\ +\infty & \text{otherwise.} \end{cases}$$

Recall that $H_0^1(Z)$ embeds compactly in $C[0,1]$ (Sobolev-Rellich theorem) and so because of the hypothesis $H(\beta)$, $j(x(\cdot)) \in L^1(0,1)$ for every $x(\cdot) \in H_0^1(Z)$.

It is easy to see that $\varphi(t,\cdot,\lambda) \in \Gamma_0(H)$ and $\partial\varphi(t,x,\lambda) = -\frac{\partial}{\partial z}(a(t,\cdot,\lambda)\frac{\partial x}{\partial z}) + \beta(x(\cdot))$ for every $x \in H_0^1(Z) \cap H^2(Z)$ (cf. also Brézis [5, Example 5]). Note that for all $(t,\lambda) \in T \times E \operatorname{dom}\varphi(t,\cdot,\lambda) = H_0^1(Z)$.

Using the hypothesis $H(a)_1$ and Poincaré's inequality, we see that for all $\lambda \in E$

$$|\varphi(t,x,\lambda) - \varphi(s,x,\lambda)| \le \widehat{k}|t-s|\|x\|_{H_0^1(Z)} \le \widehat{k}_0|t-s|\varphi(t,x,\lambda)$$

for some $\widehat{k}_0 > 0$ (independent of (t,s,x,λ)). So we have satisfied the hypotheses $H(\varphi)$ (1) and (2). Also note that $\bigcup_{\lambda\in E}\{x \in H : \|x\|_2^2 + \varphi(t,x,\lambda) \le \theta\} \subseteq \bigcup_{\lambda\in E}\{x \in H : \|x\|_2^2 + \psi(t,x,\lambda) \le \theta\}$ where $\psi(t,x,\lambda) = \frac{1}{2}\int_0^1 a(t,z,\lambda)\frac{dx}{dz}\,dz$ if $x \in H_0^1(Z)$ and $+\infty$ otherwise (recall that $j \ge 0$). The latter set is bounded in $H_0^1(Z)$, hence relatively compact in $L^2(Z)$. Thus we have satisfied the hypothesis $H(\varphi)$ (3). Furthermore because of the hypothesis $H(\alpha)_1$ and Theorem 13.12, p. 159 of Dal Maso [7], we have that $\varphi(t,\cdot,\lambda) \xrightarrow{M} \varphi(t,\cdot,\lambda)$ in H, when $\lambda_n \to \lambda$ in E. So we have checked the hypothesis $H(\varphi)$.

Let $\widehat{f}_1(t,x,\lambda)$, $\widehat{b}(t,\lambda)$ and $B_M(0) = U(t)$ be as in the first application. Also $Y = L^2(0,1) = H$ and define $L : T \times H \times E \times Y \to \mathbb{R}$ by $L(t,x,\lambda,u) = \frac{1}{2}\int_0^1 |x(z)-v(t,z,\lambda)|^2\,dz + \frac{1}{2}\int_0^1 |u(z)|^2\,dz$. Clearly the hypothesis $H(v)$ guarantees that $H(L)$ holds. Also let $x_0(\cdot,\lambda) \in H_0^1(Z)$.

Now rewrite (3) in the following equivalent subdifferential evolution equation form:

$$(3)' \qquad \left\{\begin{array}{c} -\dot{x}(t) \in \partial\varphi(t,x(t),\lambda) + \widehat{f}_1(t,x(t),\lambda) + \widehat{b}(t,\lambda)u(t) \text{ a.e.} \\ x(0) = \widehat{x}_0(\lambda) \quad (\text{with } \widehat{x}_0(\lambda)(\cdot) = x_0(\cdot,\lambda) \in H_0^1(Z)) \\ u(t) \in B_M(0) \text{ a.e., } u(\cdot)\text{-measurable.} \end{array}\right\}$$

Let $\widehat{J}(\lambda,u) = \frac{1}{2}\int_0^1 L(t,x(\lambda,u)(t),\lambda,u(t))\,dt$. Then $(P)_2$ is equivalent to

$$(P)_2' \qquad \inf\left[\sup_{\lambda\in E}\widehat{J}(\lambda,u) : u \in S_{B_M(0)}\right] = \sigma.$$

Finally apply Theorem 3.3 to get the desired result. $\square$

For $Z \subseteq \mathbb{R}^N$, $N > 1$ and Z having a C^2-boundary $\partial Z = \Gamma$, using the general framework of this paper, we can treat systems monitored by the following semilinear parabolic initial-boundary value problem:

$$(4) \quad \left\{ \begin{array}{c} \dfrac{\partial x}{\partial t} - \displaystyle\sum_{i,j=1}^{N} D_i(a_{ij}(t,z,\lambda), D_j x) = f(t,z,x(t,z),\lambda) + b(t,z,\lambda)u(z) \text{ a.e.} \\ x|_{T\times\Gamma} = 0,\ x(0,z) = x_0(z,\lambda) \text{ a.e. on } Z, \\ \|u(t,\cdot)\|_2 \le M \text{ a.e., } u(\cdot,\cdot)\text{-measurable.} \end{array} \right\}$$

In this case, our hypothesis on the coefficients $\{a_{ij}\}_{i,j=1}^N$ is the following:

$\underline{H(a)_2}$: $a_{ij}(\cdot,\cdot,\lambda) \in L^\infty(T, C_1(\overline{Z}))$ for all $\lambda \in E$, $a_{ij} = a_{ji}$, $c_1\|\eta\|^2 \le \sum_{i,j=1}^N a_{ij}(t,z,\lambda)\eta_i\eta_j \le c_2\|\eta\|^2$ for all $(t,z,\lambda) \in T \times Z \times E$ with $0 < c_1 \le c_2$, $\eta \in \mathbb{R}^N$, $|a_{ij}(t,z,\lambda) - a_{ij}(s,z,\lambda)| \le \widehat{k}|t-s|$ a.e. with $\widehat{k} > 0$ and if $\lambda_n \to \lambda$, then $a(t,\cdot,\lambda_n) \xrightarrow{w} a(t,\cdot,\lambda)$ in $L^2(Z)$ and $\sum_{j=1}^N D_i a(t,\cdot,\lambda_n) \to \sum_{j=1}^N D_i a_{ij}(t,\cdot,\lambda) \in H^{-1}(Z)$ for almost all $t \in T$.

In this case $H = L^2(Z)$ and $\varphi : T \times H \times E \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is defined by

$$\varphi(t,x,\lambda) = \left\{ \begin{array}{ll} \frac{1}{2}\int_Z \sum_{i,j=1}^N a_{ij}(t,z,\lambda) D_i x D_j x\, dz & \text{if } x \in H_0^1(Z) \\ +\infty & \text{otherwise.} \end{array} \right.$$

As before we can easily check that $H(\varphi)$ is satisfied. Furthermore, if $A_n(t) = \partial\varphi(t,\cdot,\lambda_n)$ and $A(t) = \partial\varphi(t,\cdot,\lambda)$ on $H_0^1(Z) \cap H^2(Z)$ (regularity theory of elliptic problems), then from Zhikov-Kozlov-Oleinik-Ngoan [16] we know that $A_n(t) \to A(t)x$ in $H^{-1}(Z)$ for every $x \in H_0^1(Z)$ and so $\varphi(t,\cdot,\lambda_n) \xrightarrow{M} \varphi(t,\cdot,\lambda)$ in $H = L^2(Z)$ (cf. Attouch [1]).

For example, we may have $E = \mathbb{N} \cup \{+\infty\}$ (the one point (Alexandroff) compactification of the discrete (hence locally compact) metric space $\mathbb{N}$) and $a_{ij}^n(t,z) = a_{ij}(t,z) + \frac{1}{2}\cos(nz_2)$. Using the Riemann-Lebesgue lemma, we can see that $\frac{1}{2}\cos(nz_2) \to 0$ in $H^{-1}(Z)$, but only weakly in $L^2(Z)$. Then $\|A_n(t)x - A(t)x\|_{H^{-1}(Z)} \to 0$ and so $\varphi(t,\cdot,\lambda_n) \xrightarrow{M} \varphi(t,\cdot,\lambda)$ in $H = L^2(Z)$.

(C) Minimax control of differential variational inequalities.

The general theoretical framework of this paper also incorporates differential inequalities, namely systems monitored by the following evolution equation

$$(4) \quad \left\{ \begin{array}{c} -\dot{x}(t) \in N_{K(t,\lambda)}(x(t)) + f_1(t,x(t),\lambda) + f_2(t,x(t),\lambda)u(t) \text{ a.e.} \\ x(0) = x_0(\lambda) \\ u(t) \in U(t) \text{ a.e., } u(\cdot)\text{-measurable.} \end{array} \right\}$$

Here $N_{K(t,\lambda)}(x)$ denotes the normal cone to the closed and convex set $K(t,\lambda)$ at the point x. Recall that $N_{K(t,\lambda)}(x) = \partial\delta_{K(t,\lambda)}(x)$, where $\delta_{K(t,\lambda)}(x) = 0$ if $x \in K(t,\lambda)$ and ∞ otherwise (the indicator function of the set $K(t,\lambda)$). So the system (4) is a particular case of (1). Assume the following:

<u>$H(K)$</u>: $K : T \times E \to P_{kc}(H)$ is a continuous multifunction such that

$$h(K(t,\lambda), K(s,\lambda)) \le \int_s^t v(\tau)\, d\tau$$

for all $(\lambda, t, s) \in E \times T \times T$, $s \le t$ and with $v(\cdot) \in L^2(T)$ (here $h(\cdot,\cdot)$ denotes the Hausdorff metric on $P_f(H)$; i.e. if $A, B \in P_f(H)$, then $h(A,B) = \max[\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)]$).

Than it is easy to see that the hypothesis $H(\varphi)$ is satisfied with $g_r(t) = V(t) = \int_0^t v(s)\, ds$, $\dot{g}_r(t) = v(t)$, $\beta = 2$, $\alpha = 0$, and $K_r = 1$. In addition, if $B \subseteq E$ is compact, because $K(t,\cdot)$ is continuous and $P_{kc}(H)$-valued, we have that $\overline{\bigcup_{\lambda\in B} K(t,\lambda)} \in P_k(H)$. So the hypothesis $H(\varphi)$ (3) holds. Finally, if $\lambda_n \to \lambda$ in E, then $K(t,\lambda_n) \xrightarrow{h} K(t,\lambda)$ (here $\xrightarrow{h}$ denotes convergence in the Hausdorff metric), and so $K(t,\lambda_n) \xrightarrow{M} K(t,\lambda)$, which by Mosco [11] implies that $\delta_{K(t,\lambda_n)}(\cdot) \xrightarrow{M} \delta_{K(t,\lambda)}(\cdot)$. So we have satisfied the hypothesis $H(\varphi)$. Therefore we can state the following theorem:

Theorem 4.3. *<u>If</u> the hypotheses $H(K)$, $H(f)$, $H(U)$, $H(L)$, H_0 and H_1 hold, <u>then</u> the problem (P) admits an optimal control.*

Evolution equations of the form (4) arise in mathematical economics in the study of resource allocation problems (see Cornet [6]) and in theoretical mechanics in the analysis of elastoplastic systems (see Moreau [10]). Furthermore, if $H = \mathbb{R}^N$ and $K(t,\lambda) = K(\lambda)$ (i.e. the constraint set is time invariant), then from Cornet [6] we know that (4) is equivalent to

$$(4)' \qquad \left\{ \begin{array}{c} -\dot{x}(t) \in \operatorname{proj}(-f_1(t,x(t),\lambda) - f_2(t,x(t),\lambda)u(t);\ T_{K(\lambda)}(x(t))) \text{ a.e.} \\ x(0) = x_0(\lambda) \\ u(t) \in U(t) \text{ a.e., } u(\cdot)\text{-measurable.} \end{array} \right\}$$

Here $\operatorname{proj}(\cdot\,; T_{K(\lambda)}(x))$ denotes the metric projection on the tangent cone $T_{K(\lambda)}(x)$ to $K(\lambda)$ at x. This projected evolution inclusion is useful in the study of systems with state constraints. In describing the effect of the constraint on the dynamics, in many cases we may assume that the velocity $\dot{x}(t)$ is projected at each time instant on the set of allowed directions towards the constraint set at the point $x(t)$. This leads us to the problem $(4)'$, which as we already mentioned is equivalent to (4). This is true for example in electrical networks with diode nonlinearities or mechanical systems with hysteresis (see Krasnosel'skii-Pokrovskii [8]).

As a simple illustration, consider the following problem defined in $\mathbb{R}^N$:

$$
(5)\qquad \left\{\begin{array}{c}
\theta_1(\lambda) \le x(t) \le \theta_2(\lambda),\ t \in T = [0, b],\ x(0) = x_0(\lambda) \\
\dot{x}(t) = f_1(t, x(t), \lambda) + f_2(t, x(t), \lambda)u(t) \text{ a.e. on} \\
T_1(\lambda) = \{s \in T : \theta_1(\lambda) < x(s) < \theta_2(\lambda)\} \\
\dot{x}(t) \ge f_1(t, x(t), \lambda) + f_2(t, x(t), \lambda)u(t) \text{ a.e. on} \\
T_2(\lambda) = \{s \in T : \theta_1(\lambda) = x(s)\} \\
\dot{x}(t) \le f_1(t, x(t), \lambda) + f_2(t, x(t), \lambda)u(t) \text{ a.e. on} \\
T_3(\lambda) = \{s \in T : \theta_2(\lambda) = x(s)\} \\
u(t) \in U(t) \text{ a.e., } u(\cdot)\text{-measurable.}
\end{array}\right\}
$$

We assume the following:

<u>$H(\theta)$</u>: $\theta_1, \theta_2 : E \to \mathbb{R}^N$ are continuous functions such that for all $\lambda \in E$, we have $\theta_1(\lambda) \le \theta_2(\lambda)$.

Set $K(\lambda) = \{h \in \mathbb{R}^N : \theta_1(\lambda) \le h \le \theta_2(\lambda)\} = [\theta_1(\lambda) + \mathbb{R}^N_+] \cap [\theta_2(\lambda) - \mathbb{R}^N_+]$. So $K(\cdot)$ is h-continuous, with values in $P_{kc}(\mathbb{R}^N)$. Recall that if $x \in \operatorname{int} K(\lambda)$, $T_{K(\lambda)}(x) = \mathbb{R}^N$ and so $N_{K(\lambda)}(x) = \{0\}$; if $x = \theta_1(\lambda)$, $T_{K(\lambda)}(x) = \mathbb{R}^N_+$ and so $N_{K(\lambda)}(x) = -\mathbb{R}^N_+$; and finally, if $x = \theta_2(\lambda)$, $T_{K(\lambda)}(x) = -\mathbb{R}^N_+$ and so $N_{K(\lambda)}(x) = \mathbb{R}^N_+$. So the system (5) is equivalent to

$$
(5)'\qquad \left\{\begin{array}{c}
-\dot{x}(t) \in N_{K(\lambda)}(x(t)) - f_1(t, x(t), \lambda) - f_2(t, x(t), \lambda)u(t) \text{ a.e.} \\
x(0) = x_0(\lambda) \\
u(t) \in U(t) \text{ a.e., } u(\cdot)\text{-measurable.}
\end{array}\right\}
$$

Thus we can state the following result concerning the problem (5), when our system is described by (5).

Theorem 4.4. *<u>If</u> the hypotheses $H(\theta)$, $H(f)$, $H(U)$, $H(L)$, H_0 and H_1 hold, <u>then</u> the problem (P) admits an optimal control.*

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