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Lacunary strong convergence with respect to a sequence of modulus functions

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Abstract. The definition of lacunary strong convergence is extended to a definition of lacunary strong convergence with respect to a sequence of modulus functions in a Banach space. We study some connections between lacunary statistical convergence and lacunary strong convergence with respect to a sequence of modulus functions in a Banach space.

Keywords: lacunary sequence, modulus function, statistical convergence, Banach space
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1. Introduction

By a lacunary sequence $\theta = (k_r)$ where $k_0 = 0$, we mean an increasing sequence of positive integers with $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio k_r/k_{r-1} will be denoted by q_r. The sequence space of lacunary strongly convergent sequences N_θ was defined by Freedman et al. [4], as follows:

$$N_\theta = \{x = (x_i) \,:\, \lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} |x_i - l| = 0 \ \text{ for some } \ l\}.$$

Let $\|x\|_\theta = \sup_r (h_r^{-1} \sum_{i\in I_r} |x_i|)$, whenever $x \in N_\theta$. Then $(N_\theta, \|.\|_\theta)$ is a BK-space. N_θ^0 denotes the subset of all sequences which are lacunary strongly convergent to zero. $(N_\theta^0, \|.\|_\theta)$ is also a BK-space.

There is a strong connection between N_θ and the sequence space $|\sigma_1|$, which is defined by

$$|\sigma_1| = \{x = (x_i) \,:\, \lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} |x_i - l| = 0 \ \text{ for some } \ l\}.$$

In the special case $\theta = (2^r)$, we have $N_\theta = |\sigma_1|$.

The well known space $\hat{c}$, the space of all almost convergent sequences was defined by Lorentz [9]. Later $[\hat{c}]$ the space of strong almost convergence was

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introduced by Maddox [10] and also independently by Freedman et al. [4]. This sequence space was defined as follows:

$$[\hat{c}] = \{x = (x_i) : \lim_{n\to\infty} n^{-1} \sum_{i=p+1}^{p+n} |x_i - l| = 0 \ \text{ uniformly in } p, \text{ for some } l\}.$$

We denote the space of all sequences which are strongly almost convergent to zero by $[\hat{c}_0]$. In [15], the spaces $[\hat{c}_0]$ and $[\hat{c}]$ were extended to $[\hat{c}_0(f)]$ and $[\hat{c}(f)]$.

Let X be a Banach space. We define $s(X)$ to be the vector space of all X-valued sequences, $l_\infty(X)$ the vector space of all bounded X-valued sequences and $c(X)$ the vector space of all convergent X-valued sequences. Thus $x = (x_i) \in l_\infty(X)$, if $\sup \|x_i\| < \infty$, where $x_i \in X$ for $i \in N$. Consequently $l_\infty(X)$ becomes a Banach space with the natural coordinatewise operations and $\|x\| = \sup_i \|x_i\|$ for $x \in l_\infty(X)$.

The notion of a modulus function was introduced by Nakano [13]. We recall that a modulus f is a function from $[0,\infty)$ to $[0,\infty)$ such that (i) $f(x) = 0$ if and only if $x = 0$, (ii) $f(x+y) \le f(x) + f(y)$ for $x, y \ge 0$, (iii) f is increasing and (iv) f is continuous from the right at 0. It follows that f must be continuous on $[0,\infty)$. Connor [2], Maddox [11], [12], Kolk [8], Pehlivan and Fisher[16] and Ruckle [19] used a modulus function to construct sequence spaces.

Now let S be the space of sequences of modulus functions $F = (f_i)$ such that $\lim_{u\to 0^+} \sup_i f_i(u) = 0$. Throughout this paper the sequence of modulus functions determined by F will be denoted by $F = (f_i) \in S$ for every $i \in N$.

The purpose of this paper is to introduce and study a concept of lacunary strong convergence with respect to a sequence of modulus functions in a Banach space.

2. Inclusion theorems

We now introduce the generalizations of the lacunary strongly convergent sequences and investigate some inclusion relations.

Definition 2.1. Let $F = (f_i)$ be a sequence of modulus functions in S. Let X be a Banach space. We define the spaces

$$N_\theta(X) = \{x = (x_i) \in s(X) : \lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} \|x_i - l\| = 0 \text{ for some } l \in X\},$$

$$N_\theta(X,F) = \{x = (x_i) \in s(X) : \lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} f_i(\|x_i - l\|) = 0 \text{ for some } l \in X\},$$

$$N_\theta^0(X,F) = \{x = (x_i) \in s(X) : \lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} f_i(\|x_i\|) = 0\}.$$

$N_\theta(X)$, $N_\theta(X,F)$ and $N_\theta^0(X,F)$ are linear spaces. We consider only $N_\theta(X,F)$. Suppose that $x_i \to l$ in $N_\theta(X,F)$, $y_i \to l'$ in $N_\theta(X,F)$ and α, γ are in C. Then there exist integers K_α and M_γ such that $|\alpha| \le K_\alpha$ and $|\gamma| \le M_\gamma$. We have

$$h_r^{-1} \sum_{i\in I_r} f_i(\|\alpha x_i + \gamma y_i - (\alpha l + \gamma l')\|) \\ \le K_\alpha h_r^{-1} \sum_{i\in I_r} f_i(\|x_i - l\|) + M_\gamma h_r^{-1} \sum_{i\in I_r} f_i(\|x_i - l'\|).$$

This implies that $\alpha x + \gamma y \to \alpha l + \gamma l'$ in $N_\theta(X, F)$. Note that if we put $f_i = f$ for $i \in N$ then $N_\theta(X, F) = N_\theta(X, f)$. We write $N_\theta(X, f) = N_\theta(X)$ for $f(x) = x$.

Proposition 2.2 ([16]). *Let f be a modulus and let $0 < \delta < 1$. Then for each $\|u\| \ge \delta$, we have $f(\|u\|) \le 2f(1)\delta^{-1}\|u\|$.*

Proof:

$$f(\|u\|) \le f(1 + [\|u\|/\delta]) \le f(1) + f([\|u\|/\delta]) \le f(1)(1 + \|u\|/\delta) \le 2f(1)\|u\|/\delta.$$

where $[\|u\|/\delta]$ denotes the integer part of $\|u\|/\delta$. □

Theorem 2.3. *Let X be a Banach space and let $F = (f_i)$ be a sequence of modulus functions in S. If $x = (x_i)$ is lacunary strongly convergent to l in X, then $x = (x_i)$ is lacunary strongly convergent to l in X with respect to F, i.e. $N_\theta(X) \subset N_\theta(X, F)$.*

Proof: Let $F = (f_i)$ be a sequence modulus functions in S and put $\sup_i f_i(1) = M$. Let $x \in N_\theta(X)$. Then we have

$$A_r(X) = h_r^{-1} \sum_{i\in I_r} \|x_i - l\| \to 0 \quad \text{as} \quad r \to \infty, \quad \text{for some} \quad l \in X.$$

Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f_i(u) < \varepsilon$ $(i \in N)$ for every u with $0 \le u \le \delta$. We can write

$$h_r^{-1} \sum_{i\in I_r} f_i(\|x_i - l\|) = h_r^{-1} \sum_{\substack{i\in I_r \\ \|x_i - l\| \le \delta}} f_i(\|x_i - l\|) + h_r^{-1} \sum_{\substack{i\in I_r \\ \|x_i - l\| > \delta}} f_i(\|x_i - l\|) \\ \le h_r^{-1}(h_r\varepsilon) + h_r^{-1} 2M\delta^{-1} h_r A_r(X),$$

by Proposition 2.2. Letting $r \to \infty$, it follows that $x \in N_\theta(X, F)$. □

Theorem 2.4. *Let X be a Banach space and $F = (f_i)$ be a sequence of modulus functions. If $\lim_{u\to\infty} \inf_i f_i(u)/u > 0$, then $N_\theta(X, F) = N_\theta(X)$.*

Proof: If $\lim_{u\to\infty} \inf_i f_i(u)/u > 0$ then there exists a number $c > 0$ such that $f_i(u) > cu$ for $u > 0$ and $i \in N$. We have $x \in N_\theta(X, F)$. Clearly

$$h_r^{-1} \sum_{i\in I_r} f_i(\|x_i - l\|) \ge h_r^{-1} \sum_{i\in I_r} c\|x_i - l\| = ch_r^{-1} \sum_{i\in I_r} \|x_i - l\|,$$

therefore $x \in N_\theta(X)$. By using Theorem 2.3 the proof is complete. $\square$

We now give an example to show that $N_\theta(X,F) \neq N_\theta(X)$ in the case when $\lim_{u\to\infty} \inf_i f_i(u)/u = 0$. Consider the sequence $f_i(x) = x^{1/(i+1)}$ $(i \geq 1,\ x > 0)$ of modulus functions. Now define $x_i = h_r v$ if $i = k_r$ for some $r \geq 1$ and $x_i = \theta$ otherwise, where $v \in X$ and $\|v\| = 1$. This yields

$$h_r^{-1} \sum_{i\in I_r} f_i(\|x_i\|) = h_r^{-1}(f_{k_r}(h_r\|v\|)) = h_r^{-1} h_r^{1/(1+k_r)} \to 0 \text{ as } r \to \infty$$

and so $x \in N_\theta(X,F)$. But

$$h_r^{-1} \sum_{i\in I_r} \|x_i\| = h_r^{-1} h_r \|v\| \to 1 \text{ as } r \to \infty$$

and so $x \notin N_\theta(X)$.

Proposition 2.5. *If $f_i = f$ for $i \in N$, then $[\hat{c}(X,f)] \subset N_\theta(X,f)$ for every lacunary sequence θ, where*
$[\hat{c}(X,f)] = \{x = (x_i) \in s(X) : \lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} f(\|x_{i+p} - l\|) = 0$, for some $l \in X$ uniformly in $p\}$.

To show that $N_\theta^0(X,f)$ strictly contains

$$[\hat{c}_0(X,f)] = \{x = (x_i) \in s(X) : \lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} f(\|x_{i+p}\|) = 0 \text{ uniformly in } p\},$$

we proceed as in [4; p. 513]. We define $x = (x_i)$ by $x_i = v$ if $k_{r-1} < i \leq k_{r-1} + [\sqrt{h_r}\,]$ for some r and $x_i = \theta$ otherwise, where $v \in X$ and $\|v\| = 1$. It follows that $x \notin [\hat{c}_0(X,f)]$. However $x \in N_\theta^0(X,f)$ since

$$h_r^{-1} \sum_{i\in I_r} f(\|x_i\|) = h_r^{-1} [\sqrt{h_r}\,] f(1) \to 0 \text{ as } r \to \infty.$$

If $f_i = f$ for $i \in N$ we can show as in [4] that $|\sigma_1(X,f)| = N_\theta(X,f)$ if and only if $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$, where $|\sigma_1(X,f)| = \{x = (x_i) \in s(X) : \lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} f(\|x_i - l\|) = 0$ for some $l \in X\}$.

Proposition 2.6. *Let X be a Banach space. Let $\theta = (k_r)$ be a lacunary sequence with $\liminf_r q_r > 1$ then for any modulus f, $|\sigma_1(X,f)| \subset N_\theta(X,f)$.*

Proof: It is enough to show that $|\sigma_1(X,f)|^0 \subset N_\theta^0(X,f)$. Suppose $\liminf_r q_r > 1$. There exists $\delta > 0$ such that $q_r = (k_r/k_{r-1}) \geq 1 + \delta$ for sufficiently large r. We have, for sufficiently large r, that $(k_r/h_r) \le (1+\delta)/\delta$ and $(h_r/k_r) \ge \delta/(1+\delta)$. Now write

$$\begin{aligned} k_r^{-1}\sum_{i=1}^{k_r} f(\|x_i\|) \ge k_r^{-1}\sum_{i\in I_r} f(\|x_i\|) &= (h_r/k_r)h_r^{-1}\sum_{i\in I_r} f(\|x_i\|) \\ &\ge (\delta/(1+\delta))h_r^{-1}\sum_{i\in I_r} f(\|x_i\|), \end{aligned}$$

from which we deduce that $|\sigma_1(X,f)|^0 \subset N_\theta^0(X,f)$ for any modulus f. □

Proposition 2.7. *Let X be a Banach space. Let $\theta = (k_r)$ be a lacunary sequence with $\limsup_r q_r < \infty$ then for any modulus f, $N_\theta(X,f) \subset |\sigma_1(X,f)|$.*

Proof: Let $x \in N_\theta^0(X,f)$ and $\varepsilon > 0$. There exists j_0 such that for every $j \ge j_0$

$$H_j = h_j^{-1}\sum_{i\in I_j} f(\|x_i\|) < \varepsilon.$$

We can also find $M > 0$ such that $H_j \le M$ for all j. If $\limsup_r q_r < \infty$ then there exists $B > 0$ such that $q_r < B$ for every r. Now let n be any integer with $k_{r-1} < n \le k_r$. Then

$$\begin{aligned} n^{-1}\sum_{i=1}^{n} f(\|x_i\|) \le k_{r-1}^{-1}\sum_{i=1}^{k_r} f(\|x_i\|) &= k_{r-1}^{-1}\Big\{\sum_{i\in I_1} f(\|x_i\|) + \ldots + \sum_{i\in I_r} f(\|x_i\|)\Big\} \\ &= k_{r-1}^{-1}\Big\{\sum_{j=1}^{j_0}\sum_{i\in I_j} f(\|x_i\|) + \sum_{j=j_0+1}^{r}\sum_{i\in I_j} f(\|x_i\|)\Big\} \\ &\le k_{r-1}^{-1}\sum_{j=1}^{j_0}\sum_{i\in I_j} f(\|x_i\|) + \varepsilon(k_r - k_{j_0})k_{r-1}^{-1} \\ &= k_{r-1}^{-1}\{h_1H_1 + h_2H_2 + \ldots + h_{j_0}H_{j_0}\} + \varepsilon(k_r - k_{j_0})k_{r-1}^{-1} \\ &\le k_{r-1}^{-1}\big(\sup_{1\le i\le j_0} H_i\big)k_{j_0} + \varepsilon(k_r - k_{j_0})k_{r-1}^{-1} < Mk_{r-1}^{-1}k_{j_0} + \varepsilon B \end{aligned}$$

which yields that $x \in |\sigma_1(X,f)|^0$. □

The next result follows from Proposition 2.6 and 2.7.

Theorem 2.8. *Let $\theta = (k_r)$ be a lacunary sequence with $1 < \liminf_r q_r \le \limsup_r q_r < \infty$. Then $|\sigma_1(X,f)| = N_\theta(X,f)$. In particular we have $N_{2^r}(X,f) = |\sigma_1(X,f)|$.*

3. Some results on X-lacunary statistical convergence

We now introduce natural relationship between lacunary strong convergence with respect to a sequence of modulus functions in Banach space and lacunary statistical convergence in a Banach space. In [3], Fast introduced the idea of statistical convergence, which is closely related to the concept of natural density or asymptotic density of subsets of the positive integers N. These ideas were later studied in [1], [5], [17] and [18]. If K is a subset of the positive integers N, then K_n denotes the set $\{k \in K : k \leq n\}$ and $|K_n|$ denotes cardinality of K_n. The natural density of K is given by $\delta(K) = \lim_{n\to\infty} n^{-1}|K_n|$, see [14]. A sequence $x = (x_i)$ is statistically convergent to l if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} n^{-1}|K(\varepsilon)| = 0,$$

where $K(\varepsilon) = \{i \in N : |x_i - l| \geq \varepsilon\}$ and $|K(\varepsilon)|$ denotes cardinality of $K(\varepsilon)$. The set of all statistically convergent sequences is denoted by St.

Recently Fridy and Orhan [6], [7] introduced the following definition of lacunary statistical convergence.

Definition 3.1. Let θ be a lacunary sequence. Then a sequence $x = (x_i)$ is said to be lacunary statistically convergent to a number l if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} h_r^{-1}|K_\theta(\varepsilon)| = 0,$$

where $K_\theta(\varepsilon) = \{i \in I_r : |x_i - l| \geq \varepsilon\}$. The set of all lacunary statistically convergent sequences is denoted by St_θ.

Some results on St_θ-convergence and St-convergence were given in [7]. It was shown there that $St = St_\theta$ if and only if $1 < \lim_r \inf q_r \leq \lim_r \sup q_r < \infty$.

Definition 3.2. Let θ be a lacunary sequence. Then a sequence $x = (x_i) \in s(X)$ is said to be X-lacunary statistically convergent to an $l \in X$ if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} h_r^{-1}|\{i \in I_r : \|x_i - l\| \geq \varepsilon\}| = 0.$$

The set of all such sequences $x = (x_i)$ is denoted by $St_\theta(X)$.

In the next section we establish inclusion relations between $St_\theta(X)$ and $N_\theta(X, F)$.

Theorem 3.3. *Let $F = (f_i)$ be a sequence of modulus functions in S. Let X be a Banach space. Then $N_\theta(X, F) \subset St_\theta(X)$ if and only if* $\inf_i f_i(u) > 0$, $(u > 0)$.

Proof: If $\inf_i f_i(u) > 0$ then there exists a number $\alpha > 0$ such that $f_i(u) \geq \alpha$ for $u > 0$ and $i \in N$. Let $x \in N_\theta(X, F)$, $\varepsilon > 0$ and $K_\theta(X, \varepsilon) = \{i \in I_r : \|x_i - l\| \geq \varepsilon\}$ then

$$h_r^{-1} \sum_{i\in I_r} f_i(\|x_i - l\|) \geq h_r^{-1} \sum_{i\in K_\theta(X,\varepsilon)} f_i(\|x_i - l\|) \geq \alpha h_r^{-1}|K_\theta(X, \varepsilon)|$$

and it follows that $x \in St_\theta(X)$.

Conversely we can select subsequence k_{r_j} of the lacunary sequence and choose a number $z \geq \varepsilon > 0$ such that $f_i(z) = 0$ for $i \in I_{r_j}$. Now define a sequence $x = (x_i)$ by putting $x_i = zv$ if $i \in I_{r_j}$ for some $j = 1, 2, \ldots$ and $x_i = \theta$ otherwise, where $v \in X$ and $\|v\| = 1$. Then we have $x \in N_\theta(X, F)$ but $x \notin St_\theta$.

Theorem 3.4. *Let $F = (f_i)$ be a sequence of modulus functions in S. Let X be a Banach space. Then $St_\theta(X) \subset N_\theta(X, F)$ if and only if $\sup_u \sup_i f_i(u) < \infty$.*

Proof: We suppose $T(u) = \sup_i f_i(u)$ and $T = \sup_u T(u)$. Let $x \in St_\theta(X)$. Since $f_i(u) \leq T$ for $i \in N$ and $u > 0$, we have

$$h_r^{-1} \sum_{i \in I_r} f_i(\|x_i - l\|) = h_r^{-1} \Big\{ \sum_{\substack{i \in I_r \\ \|x_i - l\| \geq \varepsilon}} f_i(\|x_i - l\|) + \sum_{\substack{i \in I_r \\ \|x_i - l\| < \varepsilon}} f_i(\|x_i - l\|) \Big\}$$

$$\leq h_r^{-1} \Big\{ T |\{i \in I_r \,:\, \|x_i - l\| \geq \varepsilon\}| + h_r T(\varepsilon) \Big\}.$$

Taking the limit as $\varepsilon \to 0$, it follows that $x \in N_\theta(X, F)$, proving the sufficiency.

Conversely, suppose that $\sup_u \sup_i f_i(u) = \infty$. Then we have $0 < u_1 < u_2 < \ldots < u_{r-1} < u_r < \ldots$ such that $f_{k_r}(u_r) \geq h_r$ for $r \geq 1$. We define the sequence $x = (x_i)$ by $x_i = u_r v$ if $i = k_r$ for some $r = 1, 2, \ldots$ and $x_i = \theta$ otherwise, where $v \in X$ and $\|v\| = 1$. We have $x \in St_\theta(X)$ but $x \notin N_\theta(X, F)$. □

Corollary 3.5. *Let $F = (f_i)$ be a sequence of modulus functions in S and let X be a Banach space. Then $N_\theta(X, F) = St_\theta(X)$ if and only if $\inf_i f_i > 0$ and $\sup_u \sup_i f_i(u) < \infty$. In particular, if $f_i = f$ is a modulus function, we have $N_\theta(X, f) = St_\theta(X)$ if and only if f is bounded.*

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