The C^1 stability of slow manifolds for a system of singularly perturbed evolution equations

Daniel Ševčovič

Abstract. In this paper we investigate the singular limiting behavior of slow invariant manifolds for a system of singularly perturbed evolution equations in Banach spaces. The aim is to prove the C^1 stability of invariant manifolds with respect to small values of the singular parameter.

Keywords: singularly perturbed evolution equations, C^1 stability of inertial manifolds

Classification: 35C30, 35B25, 34E15, 35B40, 35L15

1. Introduction

In this paper we consider the following system of singularly perturbed evolution equations

$$(1)_\varepsilon \qquad \begin{aligned} u_t + A_\varepsilon u &= G_\varepsilon(u,v) \\ \varepsilon v_t + B_\varepsilon v &= F_\varepsilon(u,v) \end{aligned}$$

where $\varepsilon \geq 0$ is a small parameter, $\{A_\varepsilon\}_{\varepsilon\geq 0}$, $\{B_\varepsilon\}_{\varepsilon \geq 0}$ are continuously depending families of sectorial operators in Banach spaces X and Y, respectively; $G_\varepsilon : X^\alpha \times Y^\beta \to X$, $F_\varepsilon : X^\alpha \times Y^\beta \to Y$, $\alpha, \beta \in [0,1)$; are smooth and bounded functions, $G_\varepsilon \to G_0$, $F_\varepsilon \to F_0$ as $\varepsilon \to 0^+$.

In the qualitative analysis of evolutionary differential equations, the theory of invariant manifolds plays an important rôle. It is well known that the proof of existence of center-unstable invariant manifolds carries over from the ODE setting to abstract semilinear evolution equations in Banach spaces (see, e.g. Chow & Lu [1] and references therein). Under suitable assumptions on the spectrum of A_ε and B_ε it has been proven that the dynamics of solutions of (1) resembles the behavior of a a dynamical system generated by some ODE when restricted on so-called inertial form (Foias, Sell & Temam [2]). Such inertial manifolds are even shown to be C^k smooth embedded submanifolds of the phase-space, provided that the nonlinearities G_ε, F_ε are of the same regularity class (Chow & Lu [1]). In fact, they are usually constructed as a C^k smooth graph over some finite dimensional space.

The aim of this paper is to investigate the singular limiting behavior of invariant manifolds for the system (1) when $\varepsilon \to 0^+$. More precisely, the question to be considered below is whether the inertial manifold $\mathcal{M}_\varepsilon$ for $(1)_\varepsilon$, $0 < \varepsilon \ll 1$, is close in the C^1 topology to the inertial manifold $\mathcal{M}_0$ corresponding to the quasidynamic approximation of $(1)_\varepsilon$, $\varepsilon = 0$, i.e.

$$(1)_0 \qquad \begin{aligned} u_t + A_0 u &= G_0(u,v) \\ B_0 v &= F_0(u,v)\,. \end{aligned}$$

In the geometric singular perturbation theory, such a manifold of solutions is referred to as a slow manifold. We prove the existence of an inertial manifold $\mathcal{M}_\varepsilon$ for the perturbed system $(1)_\varepsilon$, $0 < \varepsilon \ll 1$, as well as the inertial manifold $\mathcal{M}_0$. Such an invariant manifold is constructed as a graph of a C^1 smooth function, i.e. $\mathcal{M}_\varepsilon = \mathrm{Graph}(\Phi_\varepsilon)$. The main goal is to show that $\Phi_\varepsilon \to \Phi_0$ in a C^1 sense. The invariant manifolds are shown to be exponentially attractive and the semiflow $\mathcal{S}_\varepsilon$ when restricted to the manifold $\mathcal{M}_\varepsilon$ is generated by solutions of the inertial form which is an ODE

$$(2) \qquad p_t = \hat{G}_\varepsilon(p), \quad p \in E^m$$

in the Euclidean space E^m. The main result of this paper (Theorem 8) implies that the vector field $\tilde{G}_\varepsilon : E^m \to E^m$ is continuous at $\varepsilon = 0$ with respect to the $C^1(E^m, E^m)$ topology. Therefore such a result can be a useful tool, e.g. in the local bifurcation analysis when one is interested in extension of various bifurcation phenomena arising in the reduced system to the perturbed system of governing evolution equations.

In order to construct an inertial manifold $\mathcal{M}_\varepsilon$, $\varepsilon \geq 0$, we follow the classical Lyapunov-Perron method of integral equations. We first treat the singularly perturbed equation $\varepsilon v_t + B_\varepsilon v = F_\varepsilon(u,v)$ and we show that there is a nonlocal solution operator $v = \phi_\varepsilon(u)$ acting on a Banach scale of functional spaces consisting of all globally defined solutions of this equation. It should be emphasized that the derivative $D\phi_\varepsilon$ becomes continuous with respect to $\varepsilon \to 0^+$ only when ϕ_ε operates in a subclass of Hölder continuous curves. By contrast to the usual choice of a functional space (e.g. Chow & Lu [1], Foias, Sell & Temam [2], Marion [4] or Miklavčič [5]), our setting involves scales of spaces of Hölder continuous curves growing exponentially at $-\infty$. The Hölder exponent depends merely on $\alpha \in [0,1)$. In order to prove the existence of an inertial manifold $\mathcal{M}_\varepsilon = \mathrm{Graph}(\Phi_\varepsilon)$, $0 \leq \varepsilon \ll 1$, for the semiflow $\mathcal{S}_\varepsilon$ generated by solutions of $(1)_\varepsilon$ (cf. [1]) we set up an integral equation for the nonlocal equation $u_t + A_\varepsilon u = G_\varepsilon(u, \phi_\varepsilon(u))$. We then show the convergence $\Phi_\varepsilon \to \Phi_0$ in the C^1 topology. To this end, we apply a two parameter contraction principle due to Mora & Sola-Morales [6] covering differentiability and continuity of a family of nonlinear mappings operating in a scale of Banach spaces.

The methods used in the proof of the main theorem are similar, in spirit and technique, to those of the paper [8] where the author has studied the problem of C^1 smoothness of the singular limit of finite dimensional invariant manifolds in the case when the nonlinearity F depends on the u-variable only. The last assumption makes the analysis of the singularly perturbed equation considerably easier. Moreover, the exponential attractivity of invariant manifolds has not been proven in [8], and the results obtained in [8] cannot be applied to some problems arising e.g. in the theory of so-called Sobolev's equations.

The outline of this paper is as follows: Section 2 is devoted to preliminaries. We introduce the notion of a scale of Banach spaces of Hölder continuous curves parametrized by their growth at $-\infty$. We also recall some useful results regarding properties of a family of sectorial operators. In Section 3 we are interested in the problem of the existence, C^1 smoothness and continuity w.r. to $\varepsilon \to 0^+$ of a family of inertial manifolds $\mathcal{M}_\varepsilon$ for the system $(1)_\varepsilon$, $0 \le \varepsilon \ll 1$. The main result of this paper is contained in Theorem 8. As an example we consider the following equation of Sobolev type

$$(A - \mu^{(\xi)})w_t + A^2 w = f(w),$$

where $\mu^{(\xi)} \to \mu^{(0)}$ as $\xi \to 0^+$. In the case of resonance, i.e. Ker $(A - \mu^{(0)}) \neq 0$ and Ker $(A - \mu^{(\xi)}) = 0$, $0 < \xi \ll 1$, the aim is to show, under suitable assumptions on A, that the semiflow generated by the above equation is C^1 stable in the singular limit $\xi \to 0^+$.

2. Preliminaries

Let $\mathcal{X}$ be a Banach space. For any $\mu > 0$ we denote the Banach space

$$C_\mu^-(\mathcal{X}) := \left\{ u : C(R^-, \mathcal{X}), \text{ and } \|u\|_{C_\mu^-(\mathcal{X})} := \sup_{t \le 0} e^{\mu t} \|u(t)\|_{\mathcal{X}} < \infty \right\}.$$

For any $\varrho \in (0,1]$, $a \in (0,1]$ and $\mu > 0$ we furthermore introduce the Banach space $C_{\mu,\varrho,a}^-(\mathcal{X})$ of Hölder continuous curves growing exponentially at $-\infty$,

$$C_{\mu,\varrho,a}^-(\mathcal{X}) = \\ = \left\{ u \in C_\mu^-(\mathcal{X}); [u]_{\mu,\varrho,a} = \sup_{t \le 0, h \in (0,a]} \frac{\|e^{\mu t} u(t) - e^{\mu(t-h)} u(t-h)\|}{h^\varrho} < \infty \right\}$$

endowed with the norm $\|u\|_{C_{\mu,\varrho,a}^-(\mathcal{X})} := \|u\|_{C_\mu^-(\mathcal{X})} + [u]_{\mu,\varrho,a}$. The space $C_\mu^-(\mathcal{X})$ is continuously embedded into $C_\nu^-(\mathcal{X})$, $\nu > \mu$, through a linear embedding operator

$$J_{\mu,\nu} : C_\mu^-(\mathcal{X}) \to C_\nu^-(\mathcal{X}) \tag{3}$$

with norm $\|J_{\mu,\nu}\| = 1$. At the same time, the operator $J_{\mu,\nu}$ when restricted to $C_{\mu,\varrho,a}^-(\mathcal{X})$, $J_{\mu,\nu} : C_{\mu,\varrho,a}^-(\mathcal{X}) \to C_{\nu,\varrho,a}^-(\mathcal{X})$ is again an embedding, its norm is less or equal to $\max(1, (\nu - \mu)a^{1-\varrho})$ (see [8]). Hence the families $\{C_\mu^-(\mathcal{X})\}_{\mu > 0}$ as well as $\{C_{\mu,\varrho,a}^-(\mathcal{X})\}_{\mu > 0}$ form scales of Banach spaces.

As usual, for Banach spaces E_1, E_2 and $\eta \in (0,1]$ we denote $L(E_1, E_2)$ the Banach space of all linear bounded mappings from E_1 to E_2, $C^1_{bdd}(E_1, E_2)$ the Banach space consisting of the mappings $F : E_1 \to E_2$ which are Fréchet differentiable and such that F, DF are bounded and uniformly continuous, the norm being given by $\|F\|_1 := \sup|F| + \sup|DF|$. $C^{1+\eta}_{bdd}(E_1, E_2)$ will denote the Banach space consisting of the mappings $F \in C^1_{bdd}(E_1, E_2)$ such that DF is η-Hölder continuous, the norm being given by $\|F\|_{1,\eta} := \|F\|_1 + \sup_{x \neq y} \|DF(x) - DF(y)\| \|x - y\|^{-\eta}$. If $F : E_1 \to E_2$ is a bounded and Lipschitz continuous mapping, then the Nemitzky operator

$$\tilde{F} : C^-_\mu(E_1) \to C^-_\mu(E_2), \qquad \tilde{F}(u)(t) := F(u(t))$$

is bounded and Lipschitzian as well, $\sup|\tilde{F}| \le \sup|F|$ and $\mathrm{Lip}(\tilde{F}) \le \mathrm{Lip}(F)$. Let us emphasize the known fact: if $F \in C^1_{bdd}(E_1, E_2)$ then the mapping $\tilde{F} : C^-_\mu(E_1) \to C^-_\mu(E_2)$ need not be necessarily differentiable. However, it becomes C^1 smooth after composition with the embedding operator $J_{\mu,\nu}$, $\nu > \mu$,

Lemma 1 ([12, Lemma 5], [8, Lemma 2.1])**.** *Assume $F \in C^1_{bdd}(E_1, E_2)$. Then, for any $\nu > \mu > 0$, we have $\tilde{F} \in C^1_{bdd}(C^-_\mu(E_1), C^-_\nu(E_2))$ and $\tilde{F} \in C^1_{bdd}(C^-_{\mu,\varrho,a}(E_1), C^-_\nu(E_2))$, the derivative being given by $D\tilde{F}(u)h = J_{\mu,\nu} d\tilde{F}(u)h$ where $d\tilde{F}(u)h = DF(u(.))h(.)$.*

In what follows we recall some useful perturbation results for a family of sectorial operators (see [8, Section 2]). Let $\{A_\varepsilon\}_{\varepsilon \ge 0}$ be a family of closed densely defined linear operators in a Banach space X. Consider the following hypotheses:

$$\text{(H1)} \quad \begin{cases} D(A_0) = D(A_\varepsilon) \ \text{ and } \ A_0^{-1} A_\varepsilon^{-1} = A_\varepsilon^{-1} A_0^{-1}, \ \varepsilon \in [0, \varepsilon_0]; \\ 0 \in \varrho(A_\varepsilon), \ \varepsilon \in [0, \varepsilon_0], \ \text{ and } \ A_0 A_\varepsilon^{-1} \to I \ \text{ as } \ \varepsilon \to 0^+ \text{ in } L(X, X); \\ A_0 \ \text{ is a sectorial operator in } X \text{ and } \ \mathrm{Re}\, \sigma(A_0) > \omega > 0. \end{cases}$$

We refer to [3, Chapter 1] for the definition of a sectorial operator. According to [8, Lemma 2.1] the operator A_ε is also sectorial in X and $\mathrm{Re}\, \sigma(A_\varepsilon) > \omega > 0$ for any $\varepsilon > 0$ sufficiently small. Besides the hypotheses (H1) we also impose the assumptions:

$$\text{(H2)} \quad \begin{cases} A_0^{-1} : X \to X \ \text{ is a compact linear operator;} \\ \\ \text{there are } \ 0 < \lambda_- < \lambda_+ < \infty \ \text{ such that } \ \sigma(A_0) = \sigma^0_- \cup \sigma^0_+ \ \text{ where} \\ \sigma^0_\pm = \{\lambda \in \sigma(A_0); \ \ \mathrm{Re}\, \lambda \gtrless \lambda_\pm\} \end{cases}$$

Under the assumptions (H1) and (H2) we have

$$\sigma(A_\varepsilon) = \sigma^\varepsilon_- \cup \sigma^\varepsilon_+, \qquad \text{for any } \ 0 < \varepsilon \ll 1 \ \text{ small,}$$

where $\sigma_{\pm}^{\varepsilon} = \{\lambda \in \sigma(A_\varepsilon);\ \ \mathrm{Re}\,\lambda \gtrless \lambda_{\pm}\}$ (cf. [8, Lemma 2.2]). Denote by $P_\varepsilon : X \to X$ the projector associated with the linear operator A_ε and the spectral set σ_{-}^{ε}. We also denote $Q_\varepsilon := I - P_\varepsilon$; $A_{1,\varepsilon} := P_\varepsilon A_\varepsilon$; $A_{2,\varepsilon} := Q_\varepsilon A_\varepsilon$ and let

$$X_{1,\varepsilon} := P_\varepsilon X, \qquad X_{2,\varepsilon} := Q_\varepsilon X,$$

be the complementary subspaces invariant with respect to A_ε. Since $A_0^{-1} : X \to X$ is assumed to be compact and A_0 is a sectorial operator we conclude that the set σ_{-}^{0} is finite. With regard to [8, Lemma 2.2], we have $P_\varepsilon \to P_0$ as $\varepsilon \to 0^+$. Hence $P_\varepsilon|_{X_{1,0}} : X_{1,0} \to X_{1,\varepsilon}$ is a linear isomorphism, $\dim X_{1,\varepsilon} = \dim X_{1,0} < \infty$ and there exists an inverse operator

$$P_\varepsilon^{(-1)} := \left(P_\varepsilon|_{X_{1,0}}\right)^{-1} : X_{1,\varepsilon} \to X_{1,0} \tag{4}$$

of the projector P_ε restricted to $X_{1,0}$ (see [8, Lemma 2.2]). Further, $P_\varepsilon^{(-1)} P_\varepsilon \to I$ as $\varepsilon \to 0^+$ in the space $L(X_{1,0}, X_{1,0})$.

If A is a sectorial operator then $-A$ generates an analytic semigroup of linear operators $\exp(-At), t \geq 0$. If $\mathrm{Re}\,\sigma(A) > 0$ then the fractional power operator $A^\alpha, \alpha \in R$, can be defined (see e.g. [3]). As the spectral set σ_{-}^{ε} is bounded the operator $A_{1,\varepsilon}$ is a bounded linear operator on X and hence $\exp(-A_{1,\varepsilon} t)$ can be extended to a group of operators on X, $t \in R$. The operator $A_{2,\varepsilon}$ is sectorial as well. Suppose that a family $\{A_\varepsilon\}_{\varepsilon \geq 0}$ fulfills the hypotheses (H1). Then, by [8, Lemma 2.5], there are constants $M_0 > 1$ and $\varepsilon_0 > 0$ such that, for any $\varepsilon \in [0, \varepsilon_0]$,

$$\begin{aligned} \|\exp(-A_\varepsilon t)\| &\leq M_0 e^{-\omega t}; && t \geq 0 \\ \|A_0^\alpha \exp(-A_\varepsilon t)\| &\leq M_0 t^{-\alpha} e^{-\omega t}; && t > 0. \end{aligned} \tag{5}$$

Henceforth, we will suppose that the families $\{A_\varepsilon\}_{0 \leq \varepsilon \leq \varepsilon_0}$ and $\{B_\varepsilon\}_{0 \leq \varepsilon \leq \varepsilon_0}$ fulfill the hypotheses (H1)–(H2) and (H1) in the Banach spaces X and Y, respectively. Denote

$$X^\alpha = [D(A_0^\alpha)]; \quad Y^\beta = [D(B_0^\beta)]; \qquad \alpha, \beta \in [0, 1)$$

the fractional power spaces endowed with graph norms of A_0^α and B_0^β, i.e. $\|u\|_{X^\alpha} = \|A_0^\alpha u\|$, $\|v\|_{Y^\beta} = \|B_0^\beta v\|$ (cf. [3, Chapter 1]).

Now, using the estimates (5) one can easily follow the lines of the proofs of global existence and continuity of solutions of abstract semilinear evolution equations due to Henry [3, Theorems 3.3.3, 3.3.4] in order to prove that the system $(1)_\varepsilon$, $0 < \varepsilon \leq \varepsilon_0$, generates a semiflow $\mathcal{S}_\varepsilon(t)$, $t \geq 0$, defined by solutions of $(1)_\varepsilon$ on the phase-space $X^\alpha \times Y^\beta$. By a global solution of $(1)_\varepsilon$ with the initial condition $(u_0, v_0) \in X^\alpha \times Y^\beta$ we understand a function $(u, v) \in C_{loc}([0, \infty); X^\alpha \times Y^\beta) \cap C^1_{loc}((0, \infty); X^\alpha \times Y^\beta)$ such that $(u(t), v(t)) \in D(A_\varepsilon) \times D(B_\varepsilon)$, $t > 0$ and (u, v) solves the system $(1)_\varepsilon$ on $(0, \infty)$ (cf. [8, Section 3]).

In case the function $F_0 \in C^1_{bdd}(X^\alpha \times Y^\beta, Y)$ obeys the condition $\|B_0^{\beta-1}\| \sup \|D_v F_0\| < 1$, there exists a C^1_{bdd} function $\phi_0 : X^\alpha \to Y^\beta$ such that $B_0 v = F_0(u,v)$ iff $v = \phi_0(u)$. By a solution of $(1)_0$ with the initial condition $u_0 \in X^\alpha$ we understand a function $u \in C_{loc}([0,\infty); X^\alpha) \cap C^1_{loc}((0,\infty); X^\alpha)$ such that $u(t) \in D(A_0)$, $t > 0$ and u solves the equation $u_t + A_0 u = G_0(u, \phi_0(u))$ on $(0,\infty)$. Again due to the above references to Henry's lecture notes it follows that the system $(1)_0$ generates a semiflow $\tilde{\mathcal{S}}_0(t)$, $t \geq 0$, on X^α. The semiflow $\tilde{\mathcal{S}}_0$ can be naturally extended to a semiflow $\mathcal{S}_0$ acting on the manifold $\{(u, \phi_0(u)), u \in X^\alpha\}$ by $\mathcal{S}_0(t)(u, \phi_0(u)) := (\tilde{\mathcal{S}}_0(t)u, \phi_0(\tilde{\mathcal{S}}_0(t)u))$ for any $u \in X^\alpha$. Henceforth, we will identify the semiflow $\tilde{\mathcal{S}}_0$ with its extension $\mathcal{S}_0$.

Let $\mathcal{S}(t)$, $t \geq 0$, be a semiflow in the Banach space $\mathcal{X}$. We say that the set $\mathcal{M} \subset \mathcal{X}$ is an inertial manifold for the semiflow $\mathcal{S}$ if: (1) it is an invariant finite dimensional submanifold of $\mathcal{X}$; and (2) $\mathcal{M}$ attracts exponentially all solutions, i.e. there is a $\mu > 0$ such that dist $(\mathcal{S}(t)u_0, \mathcal{M}) = O(e^{-\mu t})$ as $t \to \infty$ for any $u_0 \in \mathcal{X}$ (cf. [2]).

3. Existence and the $\mathbf{C^1}$ stability of inertial manifolds

First, we will be concerned with solutions of the linear nonhomogeneous singularly perturbed problem

$$(6)_\varepsilon \qquad \varepsilon v_t + B_\varepsilon v = f$$

where $\varepsilon > 0$, $f \in C^-_\mu(Y)$, and solutions of the unperturbed problem

$$(6)_0 \qquad B_0 v = f$$

belonging to the space $C^-_\mu(Y^\beta)$.

Denote by $\mathcal{Y}_\nu$, $\mathcal{Y}_{\nu,\varrho,a}$ and $\mathcal{X}_{\nu,\varrho,a}$, $\nu > 0$, $0 < \varrho \leq 1$, $a \in (0,1]$, the following Banach spaces of bounded linear operators

$$(7) \qquad \begin{gathered} \mathcal{Y}_\nu = L(C^-_\nu(Y), C^-_\nu(Y^\beta)), \qquad \mathcal{Y}_{\nu,\varrho,a} = L(C^-_{\nu,\varrho,a}(Y), C^-_\nu(Y^\beta)), \\ \mathcal{X}_{\nu,\varrho,a} = L(C^-_\nu(X), C^-_{\nu,\varrho,a}(X^\alpha)) \,. \end{gathered}$$

Lemma 2 ([8, Lemma 3.1]). *Assume that the family $\{B_\varepsilon\}_{0\leq\varepsilon\leq\varepsilon_0}$ fulfills the hypothesis* (H1). *Then, for any $\varepsilon \in [0, \varepsilon_0]$, $0 < \nu < \omega\varepsilon_0^{-1}$, and $f \in C^-_\nu(Y)$ there is the unique solution $v \in C^-_\nu(Y^\beta)$ of $(6)_\varepsilon$ given by $v = L_\varepsilon f$ where*

$$L_\varepsilon f(t) = \frac{1}{\varepsilon}\int_{-\infty}^t \exp\left(-B_\varepsilon(t-s)/\varepsilon\right) f(s)\, ds, \quad \varepsilon > 0; \quad L_0 f(t) = B_0^{-1} f(t) \quad t \leq 0 \,.$$

The linear operator L_ε belongs to the spaces $\mathcal{Y}_\nu$ and $\mathcal{Y}_{\nu,\varrho,a}$, $0 < \varrho \leq 1$, and there is a $K_0 > 0$ such that $\|L_\varepsilon\|_{\mathcal{Y}_{\nu,\varrho,a}} \leq \|L_\varepsilon\|_{\mathcal{Y}_\nu} \leq K_0(\omega - \nu\varepsilon_0)^{\beta-1}$ for any $\varepsilon \in [0,\varepsilon_0]$, $0 < \nu J < \omega\varepsilon_0^{-1}$. Moreover, $L_\varepsilon \to L_0$ as $\varepsilon \to 0^+$ in the space $\mathcal{Y}_{\nu,\varrho,a}$.

According to the previous lemma, if $u \in C^-_\mu(X^\alpha)$ then any solution $v \in C^-_\mu(Y^\beta)$ of the equation $\varepsilon v_t + B_\varepsilon v = F_\varepsilon(u,v)$ can be written as $v = L_\varepsilon F_\varepsilon(u,v)$. The next lemma deals with unique solvability of such an equation.

Lemma 3. *Assume $F_\varepsilon \in C^{1+\eta}_{bdd}(X^\alpha \times Y^\beta, Y)$, $\varepsilon \in [0,\varepsilon_0]$, for some $\eta \in (0,1]$, and $F_\varepsilon \to F_0$ in $C^{1+\eta}_{bdd}(X^\alpha \times Y^\beta, Y)$ as $\varepsilon \to 0^+$. Let μ, κ be fixed and such that $0 < (1+\eta)\mu \le \kappa < \omega\varepsilon_0^{-1}$. Suppose that there is a $\delta < 1$ with the property $\|L_\varepsilon\|_{\mathcal{Y}_\mu}\|D_vF_\varepsilon(u,v)\|_{L(Y^\beta,Y)} \le \delta$ for any $u \in X^\alpha$, $v \in Y^\beta$ and $\varepsilon \in [0,\varepsilon_0]$. Then, for any $u \in C^-_\mu(X^\alpha)$ and $\varepsilon \in [0,\varepsilon_0]$, there is the unique solution $v = \phi_\varepsilon(u) \in C^-_\mu(Y^\beta)$ of the equation $v = L_\varepsilon F_\varepsilon(u,v)$ satisfying,*

(i) $\|\phi_\varepsilon(u_1) - \phi_\varepsilon(u_2)\|_{C^-_\mu(Y^\beta)} \le K_1\|u_1 - u_2\|_{C^-_\mu(X^\alpha)}$;

(ii) *$\lim_{\varepsilon\to 0^+} \phi_\varepsilon(u) = \phi_0(u)$ in $C^-_\mu(Y^\beta)$ uniformly w.r.t. to $u \in \mathcal{B}$ where $\mathcal{B}$ is arbitrary bounded subset of $C^-_{\mu,\varrho,a}(X^\alpha)$;*

(iii) *$\phi_\varepsilon \in C^1_{bdd}(C^-_\mu(X^\alpha), C^-_\kappa(Y^\beta))$, $\|\phi_\varepsilon\|_1 \le K_1$ and there is $d\phi_\varepsilon \in L(C^-_\mu(X^\alpha), C^-_\mu(Y^\beta))$ with the property $D\phi_\varepsilon = J_{\mu,\kappa}d\phi_\varepsilon$, $\|d\phi_\varepsilon\| \le K_1$;*

(iv) *$\lim_{\varepsilon\to 0^+} \phi_\varepsilon = \phi_0$ in $C^1_{bdd}(\mathcal{B}, C^-_\kappa(Y^\beta))$ for any bounded subset $\mathcal{B}$ of $C^-_{\mu,\varrho,a}(X^\alpha)$, where $K_1 := \|L_\varepsilon\|_{\mathcal{Y}_\mu}\|F_\varepsilon\|_1(1-\delta)^{-1}$.*

Proof: Under the assumption $\|L_\varepsilon\|_{\mathcal{Y}_\mu}\|D_vF_\varepsilon\|_{L(Y^\beta,Y)} \le \delta < 1$, the existence of the solution operator $v = \phi_\varepsilon(u)$ as well as its Lipschitz continuity (i) follows from the parametrized contraction principle.

Obviously, for $\varepsilon = 0$, we have $v = B_0^{-1}F_0(u,v)$ and $\|L_0\|_{\mathcal{Y}_\mu} = \|B_0^{\beta-1}\|$. To prove (ii), we first find an estimate of the norm of $\|\phi_0(u)\|_{C^-_{\mu,\varrho,a}(Y^\beta)}$ in terms of $u \in C^-_{\mu,\varrho,a}(X^\alpha)$. To this end, we put $v(t) = \phi_0(u)(t)$. Then, for any $t \le 0$, $h \in (0,a]$, we have

$$\begin{aligned} e^{\mu t}v(t) - e^{\mu(t-h)}v(t-h) &= (e^{\mu t} - e^{\mu(t-h)})B_0^{-1}F_0(u(t),v(t)) \\ &+ e^{\mu(t-h)}B_0^{-1}(F_0(u(t),v(t)) - F_0(u(t-h),v(t-h)))\,. \end{aligned}$$

Notice that

$$\begin{aligned} &\|w(t) - w(t-h)\|_E \\ &\le e^{-\mu t}\|e^{\mu t}w(t) - e^{\mu(t-h)}w(t-h)\|_E + (1 - e^{-\mu h})\|w(t-h)\|_E \\ &\le K_2 e^{-\mu t}\|w\|_{C^-_{\mu,\varrho,a}(E)}h^\varrho \end{aligned} \tag{8}$$

where E stands either for X^α or Y^β and $K_2 = K_2(\mu) > 0$ is a constant. Thus

$$\begin{aligned} \|e^{\mu t}v(t) - e^{\mu(t-h)}v(t-h)\|_{Y^\beta} &\le K_2\|u\|_{C^-_{\mu,\varrho,a}(X^\alpha)}h^\varrho \\ &+ \|B_0^{\beta-1}\|\|D_vF_0\|\|v(t) - v(t-h)\|_{Y^\beta}e^{\mu(t-h)}\,. \end{aligned}$$

Because $\|v\|_{C^-_\mu(Y^\beta)} \le \|B_0^{\beta-1}\|\|F_0\|_0$ and $\|L_0\|_{\mathcal{Y}_\mu}\|D_vF_0\| \le \delta < 1$, the above inequality yields the estimate

$$\|\phi_0(u)\|_{C^-_{\mu,\varrho,a}(Y^\beta)} \le K_2(1 + \|u\|_{C^-_{\mu,\varrho,a}(X^\alpha)})\,. \tag{9}$$

Arguing similarly as above one can show $\|F_0(u,v)\|_{C^-_{\mu,\varrho,a}(Y)} \le$ $K_2(1+\|u\|_{C^-_{\mu,\varrho,a}(X^\alpha)} + \|v\|_{C^-_{\mu,\varrho,a}(Y^\beta)})$. Hence

$$\|F_0(u,\phi_0(u))\|_{C^-_{\mu,\varrho,a}(Y)} \le K_2(1+\|u\|_{C^-_{\mu,\varrho,a}(X^\alpha)})\,. \tag{10}$$

As $\phi_\varepsilon(u) = L_\varepsilon F_\varepsilon(u,\phi_\varepsilon(u))$ we obtain

$$\begin{aligned}(1-\delta)\|\phi_\varepsilon(u)-\phi_0(u)\|_{C^-_\mu(Y^\beta)} \le{}& \|L_\varepsilon - L_0\|_{\mathcal{Y}_{\mu,\varrho,a}}\|F_0(u,\phi_0(u))\|_{C^-_{\mu,\varrho,a}(Y)}\\ &+\|L_\varepsilon\|_{\mathcal{Y}_\mu}\|F_\varepsilon(u,\phi_0(u)) - F_0(u,\phi_0(u))\|_{C^-_\mu(Y)}\,.\end{aligned}$$

By Lemma 2, (H1) and (10) we obtain $\lim_{\varepsilon\to 0^+}\phi_\varepsilon(u) = \phi_0(u)$ in $C^-_\mu(Y^\beta)$ uniformly w.r. to $u\in\mathcal{B}$ where $\mathcal{B}$ is an arbitrary bounded subset of $C^-_{\mu,\varrho,a}(X^\alpha)$.

(iii) For any $u,w\in C^-_\mu(X^\alpha)$, we denote

$$D\phi_\varepsilon(u)w := [I - L_\varepsilon D_v F_\varepsilon(u(.),\phi_\varepsilon(u)(.))]^{-1} L_\varepsilon D_u F_\varepsilon(u(.),\phi_\varepsilon(u)(.))w\,. \tag{11}$$

A straightforward calculation then yields

$$\begin{aligned}&\phi_\varepsilon(u+w) - \phi_\varepsilon(u) - D\phi_\varepsilon(u)w\\ &= R_\varepsilon[F_\varepsilon(u+w,\phi_\varepsilon(u)) - F_\varepsilon(u,\phi_\varepsilon(u)) - D_uF_\varepsilon(u,\phi_\varepsilon(u))w]\\ &\quad + R_\varepsilon[F_\varepsilon(u+w,\phi_\varepsilon(u+w)) - F_\varepsilon(u+w,\phi_\varepsilon(u))\\ &\quad - D_vF_\varepsilon(u,\phi_\varepsilon(u))(\phi_\varepsilon(u+w)-\phi_\varepsilon(u))]\\ &=: I_1 + I_2\end{aligned}$$

where

$$R_\varepsilon := [I - L_\varepsilon D_v F_\varepsilon(u(.),\phi_\varepsilon(u)(.))]^{-1} L_\varepsilon\,.$$

Obviously, $\|R_\varepsilon\|_{\mathcal{Y}_\nu} \le (1-\delta)^{-1}\|L_\varepsilon\|_{\mathcal{Y}_\nu}$ for $\nu=\mu$ or $\nu=\kappa$, $\varepsilon\in[0,\varepsilon_0]$. Furthermore, by Lemma 1 we have $\|I_1\|_{C^-_\kappa(Y^\beta)} = o(\|w\|_{C^-_\mu(X^\alpha)})$ as $\|w\|\to 0$. On the other hand, as $F_\varepsilon\in C^{1+\eta}_{bdd}$ and $0<(1+\eta)\mu\le\kappa$ we conclude

$$\begin{aligned}\|I_2\|_{C^-_\kappa(Y^\beta)} &= O(\|w\|^\eta_{C^-_\mu} + \|\phi_\varepsilon(u+w)-\phi_\varepsilon(u)\|^\eta_{C^-_\mu})\|\phi_\varepsilon(u+w)-\phi_\varepsilon(u)\|_{C^-_\mu}\\ &= o(\|w\|_{C^-_\mu})\,.\end{aligned}$$

Hence $\phi_\varepsilon\in C^1_{bdd}(C^-_\mu(X^\alpha), C^-_\kappa(Y^\beta))$; $D\phi_\varepsilon(u)w = J_{\mu,\kappa}d\phi_\varepsilon(u)w$ where $d\phi_\varepsilon(u)w$ is defined by the right-hand side of (11) and so $\|d\phi_\varepsilon\|\le\|L_\varepsilon\|_{\mathcal{Y}_\mu}\|F_\varepsilon\|_1(1-\delta)^{-1}$.

Finally, we prove the assertion (iv). Let $\mathcal{B}\subset C^-_{\mu,\varrho,a}(X^\alpha)$ be an arbitrary bounded set. With regard to (ii) it is sufficient to show the uniform convergence $D\phi_\varepsilon(u)\to D\phi_0(u)$ as $\varepsilon\to 0^+$ for $u\in\mathcal{B}$. For any $u\in C^-_{\mu,\varrho,a}(X^\alpha)$ we have

$$\begin{aligned}D\phi_\varepsilon(u) - D\phi_0(u) = {}&(R_\varepsilon - R_0)D_uF_0(u,\phi_0(u))\\ &+ R_\varepsilon[D_uF_\varepsilon(u,\phi_\varepsilon(u)) - D_uF_0(u,\phi_0(u))]\,.\end{aligned}$$

Now one can readily verify that

$$\begin{aligned}R_\varepsilon - R_0 = R_\varepsilon[D_vF_\varepsilon(u,\phi_\varepsilon(u)) - D_vF_0(u,\phi_0(u))]R_0\\ +[I - L_\varepsilon D_vF_\varepsilon(u,\phi_\varepsilon(u))]^{-1}(L_\varepsilon - L_0)(I + D_vF_0(u,\phi_0(u))R_0)\,.\end{aligned}$$

Furthermore,

$$\begin{aligned}D_vF_\varepsilon(u,\phi_\varepsilon(u)) - D_vF_0(u,\phi_0(u))\\ = D_v[F_\varepsilon(u,\phi_\varepsilon(u)) - F_0(u,\phi_\varepsilon(u))] + D_v[F_0(u,\phi_\varepsilon(u)) - F_0(u,\phi_0(u))]\,.\end{aligned}$$

Thus

$$\begin{aligned}\|D_vF_\varepsilon(u(t),\phi_\varepsilon(u)(t)) - D_vF_0(u(t),\phi_0(u)(t))\|_{L(Y^\beta,Y)}\\ \le \|F_\varepsilon - F_0\|_1 + \|F_0\|_{1,\eta}\|\phi_\varepsilon(u)(t) - \phi_0(u)(t)\|^\eta_{Y^\beta}\,.\end{aligned}$$

Because $0 < (1+\eta)\mu \le \kappa$ we obtain

$$\begin{aligned}\|D_vF_\varepsilon(u,\phi_\varepsilon(u)) - D_vF_0(u,\phi_0(u))\|_{L(C^-_\mu(Y^\beta),C^-_\kappa(Y))}\\ \le \|F_\varepsilon - F_0\|_1 + \|F_0\|_{1,\eta}\|\phi_\varepsilon(u) - \phi_0(u)\|^\eta_{C^-_\mu(Y^\beta)}\,.\end{aligned}$$

But the right-hand side of the above inequality tends to 0 as $\varepsilon \to 0^+$ uniformly w.r. to $u \in \mathcal{B}$. Similarly one has

$$\|D_uF_\varepsilon(u,\phi_\varepsilon(u)) - D_uF_0(u,\phi_0(u))\|_{L(C^-_\mu(X^\alpha),C^-_\kappa(Y))} \to 0 \quad \text{as} \quad \varepsilon \to 0^+$$

u.w.r. to $u \in \mathcal{B}$. Notice that $\|R_0D_uF_0(u,\phi_0(u))\|_{L(C^-_\mu(X^\alpha),C^-_\mu(Y^\beta))} \le K_1$ and

$$\begin{aligned}\|[I + D_vF_0(u,\phi_0(u))R_0]D_uF_0(u,\phi_0(u))\|_{L(C^-_{\mu,\varrho,a}(X^\alpha),C^-_{\kappa,\varrho,a}(Y))}\\ \le K_1(1 + \|u\|^\eta_{C^-_{\mu,\varrho,a}(X^\alpha)})\,.\end{aligned}$$

Indeed, let us denote

$$\mathcal{A}(t) := [I + D_vF_0(u(t),\phi_0(u)(t))R_0]D_uF_0(u(t),\phi_0(u)(t)), \quad t \le 0\,.$$

Then, by (8) and (9)

$$\begin{aligned}&\|\mathcal{A}(t) - \mathcal{A}(t-h)\|_{L(X^\alpha,Y)}\\ &\le K_1(\|u(t) - u(t-h)\|^\eta_{X^\alpha} + \|\phi_0(u)(t) - \phi_0(u)(t-h)\|^\eta_{Y^\beta})\\ &\le K_1e^{-\mu\eta t}h^{\eta\varrho}(1 + \|u\|^\eta_{C^-_{\mu,\varrho,a}(X^\alpha)})\,.\end{aligned}$$

As $0 < (1+\eta)\mu \le \kappa$ we obtain $\|\mathcal{A}(.)w\|_{C^-_{\kappa,\eta\varrho,a}(Y)} \le K_1 \|w\|_{C^-_{\mu,\varrho,a}(X^\alpha)}(1 + \|u\|^\eta_{C^-_{\mu,\varrho,a}(X^\alpha)})$ for any $w \in C^-_{\mu,\varrho,a}(X^\alpha)$. According to Lemma 2 it is now obvious that $D\phi_\varepsilon(u) \to D\phi_0(u)$ as $\varepsilon \to 0^+$ u.w.r. to $u \in \mathcal{B}$. The proof of Lemma 3 is complete. $\square$

We are in a position to construct an inertial manifold $\mathcal{M}_\varepsilon$ for the semiflow $\mathcal{S}_\varepsilon$ as the union of all Hölder continuous curves growing exponentially at $-\infty$, i.e.

$$\mathcal{M}_\varepsilon = \{(Y(\tau),\ \tau \in R,\ Y \in C^-_{\mu,\varrho,a}(X^\alpha \times Y^\beta),\ Y = (u,v) \ \text{ solves (1)}\} \tag{12}$$

for some $\mu > 0$, $\varrho \in (0,1)$ and $a \in (0,1]$. Since the system $(1)_\varepsilon$, $\varepsilon \ge 0$, is autonomous the invariance property of $\mathcal{M}_\varepsilon$ under the semiflow $\mathcal{S}_\varepsilon(t)$, $t \ge 0$, follows from the uniqueness of solutions of $(1)_\varepsilon$. By Lemma 3, $(u,v) \in C^-_{\mu,\varrho,a}(X^\alpha \times Y^\beta)$ is a solution of $(1)_\varepsilon$ if and only if $v = \phi_\varepsilon(u)$ and $u \in C^-_{\mu,\varrho,a}(X^\alpha)$ satisfies the equation $u_t(t) + A_\varepsilon u(t) = G_\varepsilon(u(t), \phi_\varepsilon(u)(t))$ on $(-\infty, 0]$. Suppose that $\lambda_- < \mu < \lambda_+$. According to [1, Lemma 4.2], $u \in C^-_{\mu,\varrho,a}(X^\alpha)$ is a solution of the integral equation

$$\begin{aligned} u(t) = \exp(-A_{1,\varepsilon}t)P_\varepsilon u(0) &+ \int_0^t \exp(-A_{1,\varepsilon}(t-s))P_\varepsilon G_\varepsilon(u(s), \phi_\varepsilon(u)(s))\,ds \\ &+ \int_{-\infty}^t \exp(-A_{2,\varepsilon}(t-s))Q_\varepsilon G_\varepsilon(u(s), \phi_\varepsilon(u)(s))\,ds\,. \end{aligned} \tag{13}$$

Let us define the linear operators $\mathcal{K}_\varepsilon : X_{1,0} \to C^-_{\mu,\varrho,a}(X^\alpha)$ and $\mathcal{T}_\varepsilon : C^-_\mu(X) \to C^-_{\mu,\varrho,a}(X^\alpha)$,

$$\begin{aligned} &\mathcal{K}_\varepsilon x(t) := \exp(-A_{1,\varepsilon}t)P_\varepsilon x; \quad \text{for any } x \in X_{1,0},\ t \le 0, \\ &\mathcal{T}_\varepsilon(g)(t) := \int_0^t \exp(-A_{1,\varepsilon}(t-s))P_\varepsilon g(s)\,ds \\ &\qquad + \int_{-\infty}^t \exp(-A_{2,\varepsilon}(t-s))Q_\varepsilon g(s)\,ds \ \text{ for any } \ g \in C^-_\mu(X), t \le 0 \end{aligned} \tag{14}$$

and the mapping $T_\varepsilon : X_{1,0} \times C^-_{\mu,\varrho,a}(X^\alpha) \to C^-_{\mu,\varrho,a}(X^\alpha)$ defined by the right-hand side of (13), i.e.

$$\begin{aligned} &T_\varepsilon(x,u)(t) := \mathcal{K}_\varepsilon x(t) + \mathcal{T}_\varepsilon(G_\varepsilon(u(.), \phi_\varepsilon(u)(.)))(t), \\ &\qquad t \le 0, \quad x \in X_{1,0}, \quad u \in C^-_{\mu,\varrho,a}(X^\alpha)\,. \end{aligned} \tag{15}$$

For any $0 \le \varepsilon \ll 1$ small, $P_\varepsilon|_{X_{1,0}} : X_{1,0} \to X_{1,\varepsilon}$ is a linear isomorphism. Then, for any $u(0) \in X^\alpha$ there exists the unique $x \in X_{1,0}$ such that $P_\varepsilon x = P_\varepsilon u(0)$. Now, using the invariance property of $\mathcal{M}_\varepsilon$ we can write the set $\mathcal{M}_\varepsilon$ as

$$\mathcal{M}_\varepsilon = \{(u(0), \phi_\varepsilon(u)(0)) \in X^\alpha \times Y^\beta,\ u = T_\varepsilon(x,u),\ x \in X_{1,0}\}\,. \tag{16}$$

The next lemma deals with the linear operators $\mathcal{K}_\varepsilon$ and $\mathcal{T}_\varepsilon$.

Lemma 4 ([8, Lemma 3.2]). *Suppose that $\varrho \in (0, 1-\alpha)$. Then there is a constant $C_1 > 0$ independent of $\varepsilon \in [0, \varepsilon_0]$ and $\lambda_\pm$ such that, for any $\mu \in (\lambda_-, \lambda_+)$, there exists a number $a(\lambda_\pm, \mu) \in (0, 1]$ with the property*

(i) $\mathcal{K}_\varepsilon \in L(X_{1,0}, C^-_{\mu,\varrho,a}(X^\alpha))$; $\quad \|\mathcal{K}_\varepsilon\|_{L(X_{1,0}, C^-_{\mu,\varrho,a}(X^\alpha))} \le C_1 \lambda_-^\alpha$ *and* $\mathcal{T}_\varepsilon \in \mathcal{X}_{\mu,\varrho,a}$; $\|\mathcal{T}_\varepsilon\|_{\mathcal{X}_{\mu,\varrho,a}} \le C_1 K(\lambda_-, \lambda_+, \mu, \alpha)$ *for any* $\varepsilon \in [0, \varepsilon_0]$ *and* $0 < a \le a(\lambda_\pm, \mu)$, *where*
$$K(\lambda_-, \lambda_+, \mu, \alpha) := \lambda_-^\alpha (\mu - \lambda_-)^{-1} + (2-\alpha)(1-\alpha)^{-1}(\lambda_+ - \mu)^{\alpha - 1};$$

(ii) $\mathcal{K}_\varepsilon \to \mathcal{K}_0$ *in* $L(X_{1,0}, C^-_{\mu,\varrho,a}(X^\alpha))$ *and* $\mathcal{T}_\varepsilon \to \mathcal{T}_0$ *as* $\varepsilon \to 0^+$ *in* $\mathcal{X}_{\mu,\varrho,a}$ *when* $\varepsilon \to 0^+$.

Henceforth, we will assume that $0 < \varrho < 1 - \alpha$ is fixed and the positive constants $\mu, \kappa, \varepsilon_0$ satisfy the inequality

$$\lambda_- < \mu < (1+\eta)\mu < \kappa < \lambda_+ \quad \text{and} \quad \varepsilon_0 \lambda_+ < \omega/2. \tag{17}$$

Let us define the Banach spaces $\mathcal{U}, \bar{\mathcal{U}}$ and E^m as follows

$$\mathcal{U} = C^-_{\mu,\varrho,a}(X^\alpha), \qquad \bar{\mathcal{U}} = C^-_{\kappa,\varrho,a}(X^\alpha), \qquad E^m = X_{1,0} \tag{18}$$

where $a := \min\{a(\lambda_\pm, \mu), a(\lambda_\pm, \kappa)\}$ and $m = \dim X_{1,0} < \infty$. Concerning the nonlinear functions G_ε and F_ε we will assume the following hypotheses:

$$\text{(H3)} \qquad \begin{cases} \text{there exist } \ \alpha, \beta \in [0,1) \ \text{ and } \ \eta \in (0,1) \ \text{ such that} \\ G_\varepsilon \in C^1_{bdd}(X^\alpha \times Y^\beta; X), \ F_\varepsilon \in C^{1+\eta}_{bdd}(X^\alpha \times Y^\beta, Y) \\ \text{for any } \ \varepsilon \in [0, \varepsilon_0]; \\ G_\varepsilon \to G_0, \quad F_\varepsilon \to F_0 \ \text{ as } \ \varepsilon \to 0^+ \ \text{ in the respective topologies.} \end{cases}$$

If, in addition to (H3), we suppose that F_ε satisfies the assumption of Lemma 3, i.e. there is a $0 < \delta < 1$ such that

$$\|L_\varepsilon\|_{\mathcal{Y}_\mu} \|D_v F_\varepsilon\| \le K_0(\omega - \mu\varepsilon_0)^{\beta - 1}\|D_v F_\varepsilon\| \le K_0(\omega/2)^{\beta-1}\|D_v F_\varepsilon\| \le \delta,$$

then the mapping $\mathcal{U} \ni u \mapsto T_\varepsilon(x, u) \in \mathcal{U}$ is Lipschitz continuous. By Lemma 3 (i), and Lemma 4, we have

$$\begin{aligned} &\|T_\varepsilon(x, u_1) - T_\varepsilon(x, u_2)\|_{\mathcal{U}} \\ &\quad \le \|\mathcal{T}_\varepsilon\|_{\mathcal{X}_{\mu,\varrho,a}} \|\tilde{G}_\varepsilon(u_1, \phi_\varepsilon(u_1)) - \tilde{G}_\varepsilon(u_2, \phi_\varepsilon(u_2))\|_{C^-_\mu(X)} \le \theta \|u_1 - u_2\|_{\mathcal{U}} \end{aligned} \tag{19}$$

where $\theta := C_2 K(\lambda_-, \lambda_+, \mu, \alpha)$ and $C_2 > 0$ is a constant independent of $\lambda_\pm, \mu \in (\lambda_-, \lambda_+)$ and $\varepsilon \in [0, \varepsilon_0]$. On the other hand, from Lemma 4 we obtain the estimate

$$\|T_\varepsilon(x_1, u) - T_\varepsilon(x_1, u)\|_{\mathcal{U}} \le \|\mathcal{K}_\varepsilon\|_{L(E^m, C^-_{\mu,\varrho,a}(X^\alpha))} \le Q\|x_1 - x_2\|_{E^m} \tag{20}$$

where $Q := C_1 \lambda_-^\alpha$. Henceforth, we will assume that $\lambda_\pm$ and $\mu \in (\lambda_-, \lambda_+)$ are chosen in such a way that the following inequality is fulfilled

$$\theta := C_2 K(\lambda_-, \lambda_+, \mu, \alpha) < 1. \tag{21}$$

Then the family of nonlinear mappings $T_\varepsilon(x, .) : \mathcal{U} \to \mathcal{U}$ undergoes the parametrized contraction principle and so, for any $x \in E^m$ and $\varepsilon \in [0, \varepsilon_0]$, there is the unique solution $u = u_\varepsilon(x)$ of the equation $u = T_\varepsilon(x, u)$ in $\mathcal{U}$.

Lemma 5. *Let $B \subset E^m$ be a bounded subset. Then*

$$\lim_{\varepsilon\to 0^+}\sup_{x\in B}\|T_\varepsilon(x,u_0(x)) - T_0(x,u_0(x))\|_{\mathcal{U}} = 0\,.$$

Proof: As $u_\varepsilon(x) = T_\varepsilon(x,u_\varepsilon(x))$ and $\mathcal{T}_\varepsilon$, $\mathcal{K}_\varepsilon$ and G_ε are bounded uniformly for $\varepsilon\in[0,\varepsilon_0]$, ε_0 small, we conclude that the set

$$\mathcal{B}_B := \{u_\varepsilon(x),\ x\in B,\ \varepsilon\in[0,\varepsilon_0]\} \tag{22}$$

is a bounded subset of $\mathcal{U}$. In particular, the set $\{u_0(x),\ x\in B\}$ is bounded in $\mathcal{U}$. Hence, by Lemma 3 (ii), we obtain $\lim_{\varepsilon\to0^+}\phi_\varepsilon(u_0(x)) = \phi_0(u_0(x))$ in $C^-_\mu(Y^\beta)$ uniformly w.r. to $x\in B$. Since $\sup_{(u,v)}\|G_\varepsilon(u,v)-G_0(u,v)\|_X\to 0$ and $\mathcal{T}_\varepsilon\to\mathcal{T}_0$ as $\varepsilon\to0^+$ we infer that $\lim_{\varepsilon\to0^+}\sup_{x\in B}\|T_\varepsilon(x,u_0(x))-T_0(x,u_0(x))\|_{\mathcal{U}}=0$. $\square$

In summary, we have shown that the family of mappings $T_\varepsilon(x,.)$ fulfills the following hypotheses:

$$\text{(T)}\quad\begin{cases}(1) & \text{there is a } \theta<1 \text{ with the property } \|T_\varepsilon(x,u_1)-T_\varepsilon(x,u_2)\|_{\mathcal{U}} \\ & \le\theta\|u_1-u_2\|_{\mathcal{U}} \text{ for any } x\in E^m,\ u_1,u_2\in\mathcal{U} \text{ and } \varepsilon\in[0,\varepsilon_0];\\[2ex] (2) & \text{there is a } Q<\infty \text{ such that } \|T_\varepsilon(x_1,u)-T_\varepsilon(x_2,u)\|_{\mathcal{U}}\le \\ & Q\|x_1-x_2\|_{E^m} \text{ for any } x_1,x_2\in E^m,\ u\in\mathcal{U} \text{ and } \varepsilon\in[0,\varepsilon_0];\\[2ex] (3) & \text{for any bounded open subset } B\subset E^m,\\ & \lim_{\varepsilon\to0^+}\sup_{x\in B}\|T_\varepsilon(x,u_0(x))-T_0(x,u_0(x))\|_{\mathcal{U}}=0.\end{cases}$$

The set $\mathcal{M}_\varepsilon$ can be represented in the form (16). Let us therefore define the mappings $\Psi_\varepsilon : E^m\to X^\alpha, \Phi_\varepsilon : E^m\to Y^\beta$ as follows

$$\Psi_\varepsilon(x) := u_\varepsilon(x)(0),\qquad \Phi_\varepsilon(x) := \phi_\varepsilon(u_\varepsilon(x))(0)\,. \tag{23}$$

Thus

$$\mathcal{M}_\varepsilon := \{(\Psi_\varepsilon(x),\Phi_\varepsilon(x)),\ x\in E^m\}\subset X^\alpha\times Y^\beta\,. \tag{24}$$

Since T_ε satisfies the hypotheses $(\mathrm{T})_1$, $(\mathrm{T})_2$ we know by the parametrized contraction principle that the set $\mathcal{M}_\varepsilon$ is a Lipschitz continuous graph of the mapping $E^m\ni x\mapsto(\Psi_\varepsilon(x),\Phi_\varepsilon(x))\in X^\alpha\times Y^\beta$. Furthermore, by Lemma 3 (ii), and $(\mathrm{T})_3$, we obtain the convergence $(\Psi_\varepsilon(x),\Phi_\varepsilon(x))\to(\Psi_0(x),\Phi_0(x))$ as $\varepsilon\to0^+$ u.w.r. to $x\in B$, B is an arbitrary bounded and open subset of E^m. In other words, the invariant set $\mathcal{M}_\varepsilon$ is an embedded Lipschitz submanifold of the phase space $X^\alpha\times Y^\beta$ and $\mathcal{M}_\varepsilon$ is C^0_{loc} close to $\mathcal{M}_0$ when ε is small enough.

By the next lemma we prove exponential attractivity of the invariant manifold $\mathcal{M}_\varepsilon$. It means that $\mathcal{M}_\varepsilon$ is an inertial manifold for the semiflow $\mathcal{S}_\varepsilon$.

Lemma 6. *Suppose that the numbers $K(\lambda_-,\lambda_+,\mu,\alpha)$, $\sup\|D_vF_\varepsilon\|$ and ε_0 are sufficiently small. Then* dist $(\mathcal{S}_\varepsilon(t)(u_0,v_0),\mathcal{M}_\varepsilon) = O(e^{-\mu t})$ *as $t\to\infty$ for any initial condition $(u_0,v_0)\in X^\alpha\times Y^\beta$.*

Proof: In the case $\varepsilon=0$, the statement of the lemma is contained in [1, Theorem 5.1]. Let $\varepsilon\in(0,\varepsilon_0]$ be fixed. In this case, the proof is again essentially the same as that of [1, Theorem 5.1]. A slight difference, in technique, of the proof is caused by the fact that we have assumed no hypotheses on the spectral gaps of the operator B_ε. Nevertheless, the lack of large spectral gaps for $\sigma(B_\varepsilon)$ is here compensated by the assumption on smallness of the norm of D_vF_ε. We therefore only sketch the main ideas of the proof.

Given a solution $(\bar u,\bar v)$ of (1) we are looking for a solution (u^*,v^*) lying on $\mathcal{M}_\varepsilon$ and satisfying the property: $(u,v)\in C^+_\mu(X^\alpha\times Y^\beta)$ where $u=u^*-\bar u$, $v=v^*-\bar v$ and C^+_μ is the Banach space

$$C^+_\mu(X^\alpha\times Y^\beta) := \{f\in C(R^+,X^\alpha\times Y^\beta),\ \|f\|_{C^+_\mu}=\sup_{t\ge 0} e^{\mu t}\|f(t)\|_{X^\alpha\times Y^\beta}<\infty\}\,.$$

Following the lines of the proof [1, Theorem 5.1], one easily verifies that the difference of solutions (u,v) belongs to C^+_μ, if and only if the following integral equations are satisfied:

$$\begin{aligned} u(t) &= \exp(-A_{2,\varepsilon}t)\xi_u + \int_0^t \exp(-A_{2,\varepsilon}(t-s))Q_\varepsilon g(s)\,ds\\ &\quad + \int_\infty^t \exp(-A_{1,\varepsilon}(t-s))P_\varepsilon g(s)\,ds \\ v(t) &= \exp(-B_\varepsilon t/\varepsilon)\xi_v + \frac{1}{\varepsilon}\int_0^t \exp(-B_\varepsilon(t-s)/\varepsilon)f(s)\,ds,\quad t\ge 0, \end{aligned}\tag{25}$$

for some $\xi=(\xi_u,\xi_v)\in X^\alpha_{2,\varepsilon}\times Y^\beta$ where

$$\begin{aligned} g(s) &:= G_\varepsilon(\bar u(s)+u(s),\bar v(s)+v(s)) - G_\varepsilon(\bar u(s),\bar v(s))\,,\\ f(s) &:= F_\varepsilon(\bar u(s)+u(s),\bar v(s)+v(s)) - F_\varepsilon(\bar u(s),\bar v(s))\,. \end{aligned}$$

It means that $u\in C^+_\mu$ is a fixed point of the mapping $u\mapsto\mathcal{G}(u,v,\xi)$ defined by the right-hand side of the first equation in (25). We will henceforth let $C>0$ denote any positive constant independent of $\lambda_\pm$ and μ. Analogously as in the proof of [1, Theorem 5.1] one can show that the mapping $\mathcal{G}$ is a uniform contraction in $C^+_\mu(X^\alpha)$. More precisely, there is a $C>0$ such that

$$\begin{aligned} &\|\mathcal{G}(u^1,v^1,\xi^1)-\mathcal{G}(u^2,v^2,\xi^2)\|_{C^+_\mu(X^\alpha)}\\ &\le C.K(\lambda_-,\lambda_+,\mu,\alpha)\left\{\|u^1-u^2\|_{C^+_\mu(X^\alpha)}+\|v^1-v^2\|_{C^+_\mu(Y^\beta)}\right\}\\ &+C\|\xi^1-\xi^2\|_{X^\alpha_{2,\varepsilon}\times Y^\beta}\,. \end{aligned}$$

By the parametrized contraction principle there exists the unique solution operator $h : C_\mu^+(Y^\beta) \times X_{2,\varepsilon}^\alpha \times Y^\beta \to C_\mu^+(X^\alpha)$ with the property: $u = \mathcal{G}(u,v,\xi)$ iff $u = h(v,\xi)$. Furthermore,

$$\begin{aligned} &\|h(v^1,\xi^1) - h(v^2,\xi^2)\|_{C_\mu^+(X^\alpha)} \\ &\le C.K(\lambda_-,\lambda_+,\mu,\alpha)\|v^1 - v^2\|_{C_\mu^+(Y^\beta)} + C\|\xi^1 - \xi^2\|_{X_{2,\varepsilon}^\alpha \times Y^\beta} . \end{aligned} \tag{26}$$

Hence v is a fixed point of the equation $v = \mathcal{F}(v,\xi)$ where $\mathcal{F}$ is defined by the right-hand side of the second equation in (25) with $f(s) := R(v,\xi)(s)$,

$$R(v,\xi)(s) := F_\varepsilon(\bar{u}(s) + h(v,\xi)(s), \bar{v}(s) + v(s)) - F_\varepsilon(\bar{u}(s),\bar{v}(s)),\ s \ge 0 .$$

Clearly,

$$\begin{aligned} &\|R(v^1,\xi^1) - R(v^2,\xi^2)\|_{C_\mu^+(Y)} \\ &\qquad \le \|F_\varepsilon\|_1 \|h(v^1,\xi^1) - h(v^2,\xi^2)\|_{C_\mu^+(X^\alpha)} + \|D_v F_\varepsilon\| \|v^1 - v^2\|_{C_\mu^+(Y^\beta)} . \end{aligned}$$

We remind ourselves that the numbers $K(\lambda_-,\lambda_+,\mu,\alpha)$, $\sup\|D_v F_\varepsilon\|$ and ε_0 are assumed to be sufficiently small. Then, taking into account (26) one can readily prove that the mapping $v \mapsto \mathcal{F}(v,\xi)$ is a uniform contraction w.r. to ξ. Denote $v^\xi \in C_\mu^+(Y^\beta)$ the unique solution of $v = \mathcal{F}(v,\xi)$. The mapping $\xi \mapsto v^\xi$ is Lipschitzian and so the mapping $X_{2,\varepsilon}^\alpha \times Y^\beta \ni \xi \mapsto (u^\xi, v^\xi) \in C_\mu^+(X^\alpha \times X^\beta)$; $u := h(v^\xi,\xi)$, is Lipschitz continuous as well. Now the rest of the proof is the same as that of [1, Theorem 5.1]. If we define $g(\xi) := P_\varepsilon(\bar{u}(0) + u^\xi(0))$ then the mapping $g : X_{2,\varepsilon}^\alpha \times Y^\beta \to X_{1,\varepsilon}$ is Lipschitz continuous. Recall that $(u^*(0), v^*(0)) \in \mathcal{M}_\varepsilon$ iff $u^*(0) = \Psi_\varepsilon(x)$ and $v^*(0) = \Phi_\varepsilon(x)$ for some $x \in E^m = X_{1,0}$. Hence the solution (u^*, v^*) belongs to $\mathcal{M}_\varepsilon$ iff $\xi = (\xi_u, \xi_v)$ solves the equation

$$\xi = (Q_\varepsilon(\Psi_\varepsilon(P_0 g(\xi)) - \bar{u}(0)),\ \ \Phi_\varepsilon(P_0 g(\xi)) - \bar{v}(0)\ ) . \tag{27}$$

Arguing similarly as in the proof of [1, Theorem 5.1] the right-hand side of the above equation if a contraction w.r. to $\xi \in X_{2,\varepsilon}^\alpha \times Y^\beta$ provided that $K(\lambda_-,\lambda_+,\mu,\alpha)$ is sufficiently small. Hence, under the assumptions of the lemma, there exists a solution ξ of (27). But this yields that $(u^*(t), v^*(t)) \in \mathcal{M}_\varepsilon$, $t \ge 0$, where $(u^*(0), v^*(0)) := (\bar{u}(0) + u^\xi(0), \bar{v}(0) + v^\xi(0))$ and $\|\bar{u}(t) - u^*(t)\|_{X^\alpha} + \|\bar{u}(t) - u^*(t)\|_{Y^\beta} = O(e^{-\mu t})$ as $t \to \infty$. It completes the proof of the lemma. □

The Banach space $\mathcal{U}$ is continuously embedded into $\bar{\mathcal{U}}$ through a linear embedding operator $J := J_{\mu,\kappa}$. Notice that $\|J_{\mu,\kappa}\| \le 1$ provided that the parameter $a \in (0,1]$ is sufficiently small. Denote $\bar{T}_\varepsilon := JT_\varepsilon$ and $\bar{u}_\varepsilon(x) := Ju_\varepsilon(x)$ for any $\varepsilon \in [0,\varepsilon_0]$ and $x \in E^m$. Now we can state a slightly modified version of the theorem due to Mora & Solà-Morales regarding the limiting behavior of fixed points of a two parametrized family of nonlinear mappings operating on a scale of Banach spaces. Their result covers differentiability and continuity of such mappings with respect to parameters.

Theorem 7 ([6, Theorem 5.1], [8, Theorem 3.6]). *Besides the hypothesis* (T) *we assume also that the mappings* $\bar{T}_\varepsilon : E^m \times \mathcal{U} \to \bar{\mathcal{U}}$, $\varepsilon \in [0, \varepsilon_0]$ *satisfy the following conditions:*

(1) *for any* $\varepsilon \in [0, \varepsilon_0]$, $\bar{T}_\varepsilon$ *is Fréchet differentiable with* $D\bar{T}_\varepsilon : E^m \times \mathcal{U} \to L(E^m \times \mathcal{U}, \bar{\mathcal{U}})$ *bounded and uniformly continuous and there exist mappings*

$$d_u T_\varepsilon : E^m \times \mathcal{U} \to L(\mathcal{U}, \mathcal{U});\ \bar{d}_u T_\varepsilon : E^m \times \mathcal{U} \to L(\bar{\mathcal{U}}, \bar{\mathcal{U}});\ d_x T_\varepsilon : E^m \times \mathcal{U} \to L(E^m, \mathcal{U})$$

such that

$$D_u \bar{T}_\varepsilon(x,u) = J d_u T_\varepsilon(x,u) = \bar{d}_u T_\varepsilon(x,u) J, \ \ D_x \bar{T}_\varepsilon(x,u) = J d_x T_\varepsilon(x,u)$$
$$\|d_u T_\varepsilon(x,u)\|_{L(\mathcal{U},\mathcal{U})} \le \theta,\ \|\bar{d}_u T_\varepsilon(x,u)\|_{L(\bar{\mathcal{U}},\bar{\mathcal{U}})} \le \theta,\ \|d_x T_\varepsilon(x,u)\|_{L(E^m,\mathcal{U})} \le Q$$

(2) *for any* B *bounded and open subset of* E^m, $D\bar{T}_\varepsilon(x,u) \to D\bar{T}_0(x,u)$ *as* $\varepsilon \to 0^+$ *uniformly for* $(x,u) \in \{(x, u_\varepsilon(x)),\ x \in B,\ \varepsilon \in [0, \varepsilon_0]\}$.

Then the mappings $\bar{u}_\varepsilon : E^m \to \bar{\mathcal{U}}$ *have the following properties:*

(a) *for any* $\varepsilon \in [0, \varepsilon_0]$; $\bar{u}_\varepsilon : E^m \to \bar{\mathcal{U}}$ *is Fréchet differentiable, with* $D\bar{u}_\varepsilon : E^m \to L(E^m, \bar{\mathcal{U}})$ *bounded and uniformly continuous,*

(b) *for any* B *bounded and open subset of* E^m, $D\bar{u}_\varepsilon(x) \to D\bar{u}_0(x)$ *as* $\varepsilon \to 0^+$ *uniformly with respect to* $x \in B$.

In order to apply the above theorem we define the mappings

$$d_u T_\varepsilon(x,u) := \mathcal{T}_\varepsilon \left(d_u \tilde{G}_\varepsilon(u, \phi_\varepsilon(u)) + d_v \tilde{G}_\varepsilon(u, \phi_\varepsilon(u)) d\phi_\varepsilon(u) \right), \quad d_x T_\varepsilon(x,u) := \mathcal{K}_\varepsilon,$$
$$\bar{d}_u T_\varepsilon(x,u) := \bar{\mathcal{T}}_\varepsilon \left(\bar{d}_u \tilde{G}_\varepsilon(u, \phi_\varepsilon(u)) + \bar{d}_v \tilde{G}_\varepsilon(u, \phi_\varepsilon(u)) d\phi_\varepsilon(u) \right)$$

where the linear operators $\mathcal{T}_\varepsilon \in \mathcal{X}_{\mu,\varrho,a}$, $\bar{\mathcal{T}}_\varepsilon \in \mathcal{X}_{\kappa,\varrho,a}$, $\mathcal{K}_\varepsilon \in L(E^m, C^-_{\mu,\varrho,a}(X^\alpha))$ were introduced in (14) and the linear mappings

$$\begin{array}{ll} d_u \tilde{G}_\varepsilon(u,v) \in L(\mathcal{U}, C^-_\mu(X)); & \bar{d}_u \tilde{G}_\varepsilon(u,v) \in L(\bar{\mathcal{U}}, C^-_\kappa(X)), \\ d_v \tilde{G}_\varepsilon(u,v) \in L(C^-_\mu(Y^\beta), C^-_\mu(X)); & \bar{d}_u \tilde{G}_\varepsilon(u,v) \in L(C^-_\kappa(Y^\beta), C^-_\kappa(X)), \\ d\phi_\varepsilon(u) \in L(\mathcal{U}, C^-_\mu(Y^\beta)); & \bar{d}\phi_\varepsilon(u) \in L(\mathcal{U}, C^-_\kappa(Y^\beta)) \end{array}$$

are such that $D_i \tilde{G}_\varepsilon = J_{\mu,\kappa} d_i \tilde{G}_\varepsilon$, $i = u$ or $i = v$, where $\tilde{G}_\varepsilon \in C^1_{bdd}(\mathcal{U} \times C^-_\mu(Y^\beta), C^-_\kappa(X))$ (see Lemma 1). From this we infer $D_u \bar{T}_\varepsilon(x,u) = J_{\mu,\kappa} d_u T_\varepsilon(x,u) = \bar{d}_u T_\varepsilon(x,u) J_{\mu,\kappa}$. Arguing similarly as in the proof of the estimate (19) one obtains that the family $T_\varepsilon(x,.)$ obeys the assumption (i) of Theorem 7 with the constants $Q > 0$ and $0 < \theta < 1$ given by (20) and (21), respectively.

Finally, let B be a bounded and open subset of E^m. By (22), the set $\mathcal{B}_B = \{u_\varepsilon(x),\, x\in B,\, \varepsilon\in[0,\varepsilon_0]\}$, is a bounded subset of $\mathcal{U}$. According to Lemma 3 (iii), we conclude that

$$\lim_{\varepsilon\to 0^+}\sup_{u\in\mathcal{B}_B}\|D\phi_\varepsilon(u)-D\phi_0(u)\|_{L(\mathcal{U},C^-_{(1+\eta)\mu}(Y^\beta))}=0$$

where ϕ_ε is considered as a C^1_{bdd} function from $\mathcal{U}$ into $C^-_{(1+\eta)\mu}(Y^\beta)$. Since $\tilde{G}_\varepsilon\in C^1_{bdd}(\mathcal{U}\times C^-_\nu(Y^\beta),C^-_\kappa(X))$ where ν stands either for μ or $(1+\eta)\mu$ and $\lim_{\varepsilon\to0^+}\phi_\varepsilon(u)=\phi_0(u)$ in $C^-_\nu(Y^\beta)$ u.w.r. to $u\in\mathcal{B}_B$, we obtain the convergence $D\tilde{G}_\varepsilon(u,\phi_\varepsilon(u))\to D\tilde{G}_0(u,\phi_0(u))$ as $\varepsilon\to0^+$. Therefore the derivative

$$D_u\bar{T}_\varepsilon(x,u)=\bar{\mathcal{T}}_\varepsilon\left(D_u\tilde{G}_\varepsilon(u,\phi_\varepsilon(u))+D_v\tilde{G}_\varepsilon(u,\phi_\varepsilon(u))D\phi_\varepsilon(u)\right)$$

converges towards $D_u\bar{T}_0(x,u)$ when ε tends to zero u.w.r. to $u\in\mathcal{B}_B$ and $x\in B$. Obviously, $D_x\bar{T}_\varepsilon(x,u)=J_{\mu,\kappa}\mathcal{K}_\varepsilon\to D_x\bar{T}_0(x,u)$ as $\varepsilon\to0^+$. In this way we have shown that the family of operators $\bar{T}_\varepsilon$ fulfills all the hypotheses of Theorem 7.

Therefore $u_\varepsilon\in C^1_{bdd}(E^m,\bar{\mathcal{U}})$ and, for any bounded and open subset $B\subset E^m$, we have $u_\varepsilon\to u_0$ as $\varepsilon\to0^+$ in $C^1_{bdd}(B,\bar{\mathcal{U}})$. Taking into account (22) and Lemma 3 (iv), we furthermore know that $\phi_\varepsilon(u_\varepsilon(x))\to\phi_0(u_0(x))$ in $C^1_{bdd}(B,C^-_{\bar\kappa}(Y^\beta))$ for some $\bar\kappa>\kappa$ u.w.r. to $x\in B$. Since $\Psi_\varepsilon(x)=u_\varepsilon(x)(0)$ and $\Phi_\varepsilon(x)=\phi_\varepsilon(u_\varepsilon(x))(0)$ we also infer that $(\Psi_\varepsilon,\Phi_\varepsilon)\in C^1_{bdd}(E^m,X^\alpha\times Y^\beta)$ and $(\Psi_\varepsilon,\Phi_\varepsilon)\to(\Psi_0,\Phi_0)$ as $\varepsilon\to0^+$ in the space $C^1_{bdd}(B,X^\alpha\times Y^\beta)$. Finally, we notice that the usual choice for the parameter $\mu\in(\lambda_-,\lambda_+)$ is to set $\mu:=(\lambda_-+\lambda_+)/2$.

Summarizing the above results, we are in a position to state the main theorem of this paper.

Theorem 8. *Assume that the families $\{A_\varepsilon\}_{\varepsilon\ge0}$ and $\{B_\varepsilon\}_{\varepsilon\ge0}$ satisfy the hypotheses* (H1)–(H2) *and* (H1) *in the Banach spaces X and Y, respectively. Assume that the nonlinearities G_ε and F_ε fulfill the hypothesis* (H3).

If the numbers $\lambda_-^\alpha(\lambda_+-\lambda_-)^{-1}$, $(\lambda_+-\lambda_-)^{-1}$, $\sup_{\varepsilon,u,v}\|D_vF_\varepsilon(u,v)\|$ and $\varepsilon_0>0$ are sufficiently small, then, for any $\varepsilon\in[0,\varepsilon_0]$, there exists an inertial manifold $\mathcal{M}_\varepsilon\subset X^\alpha\times Y^\beta$ for the semiflow $\mathcal{S}_\varepsilon(t)$, $t\ge0$, generated by the system $(1)_\varepsilon$. Moreover,

(a) $\dim\mathcal{M}_\varepsilon=\dim\mathcal{M}_0=m<\infty$;
(b) $\mathcal{M}_\varepsilon=\{(\Psi_\varepsilon(x),\Phi_\varepsilon(x)),\ x\in E^m\}$
where $(\Psi_\varepsilon,\Phi_\varepsilon)\in C^1_{bdd}(E^m,X^\alpha\times Y^\beta)$;
(c) *for any bounded and open subset* $B\subset E^m$,
$\lim_{\varepsilon\to0^+}(\Psi_\varepsilon,\Phi_\varepsilon)=(\Psi_0,\Phi_0)$ *in the space* $C^1_{bdd}(B,X^\alpha\times Y^\beta)$.

Remark 9. The assumption that the nonlinearities G_ε and F_ε are smoothly bounded functions is not too much restrictive in the case when we are dealing with so-called dissipative semiflows. If there exists a bounded subset $\mathcal{D}$ of the phase-space attracting any solution ($\mathcal{D}$ does not depend on ε and the phase-space admits a $C^{1+\eta}$ smooth bump function) then one can modify the original nonlinearities by zero far from the vicinity of $\mathcal{D}$ in such a way that the modified nonlinearities fulfill the hypothesis (H3). In such a case we have however constructed a local center-unstable manifold $\mathcal{M}_\varepsilon^{loc}$ instead of a global inertial manifold. The existence of such a uniform dissipative set has been verified, e.g. for a class of singularly perturbed beam equations (see [8], [9]).

Remark 10. The corresponding inertial form for $(1)_\varepsilon$ is obtained by taking P_ε projection of the first equation in (1). The resulting equation is an ODE in the finite dimensional linear space $X_{1,\varepsilon}$. With regard to (4) we then apply the linear operator $P_\varepsilon^{(-1)} : X_{1,\varepsilon} \to X_{1,0}$ to obtain an ODE in the Euclidean space $E^m = X_{1,0}$. Namely,

$$p_t = -P_\varepsilon^{(-1)} A_{1,\varepsilon} P_\varepsilon p + P_\varepsilon^{(-1)} P_\varepsilon G_\varepsilon(\Psi_\varepsilon(p), \Phi_\varepsilon(p)) =: \hat{G}_\varepsilon(p)\,.$$

Hence the dynamics on the invariant manifold $\mathcal{M}_\varepsilon$ is governed by solutions of the equation $p_t = \hat{G}_\varepsilon(p)$ in a sense that $(u, v) \subset \mathcal{M}_\varepsilon$ is a solution of $(1)_\varepsilon$ iff $u = \Psi_\varepsilon(p)$, $v = \Phi_\varepsilon(p)$ where p is a solution of the ODE $p_t = \hat{G}_\varepsilon(p)$ in E^m. The vector field $\hat{G}_\varepsilon$ belongs to the class $C^1_{bdd}(E^m, E^m)$ and, moreover, $\hat{G}_\varepsilon \to \hat{G}_0$ as $\varepsilon \to 0^+$ in the topology of $C^1_{bdd}(B, E^m)$ where $B \subset E^m$ is arbitrary bounded and open subset.

Example. We will apply the results obtained to certain resonance problem arising in the study of degenerate Sobolev's equations. Let us consider the following Sobolev equation

$$(A - \mu^{(\xi)})w_t + A^2 w = f(w) \tag{28}$$

where $A : D(A) \subset \mathcal{X} \to \mathcal{X}$ is a self-adjoint positive definite operator in a Hilbert space $\mathcal{X}$, $A^{-1} : \mathcal{X} \to \mathcal{X}$ is compact, $f \in C^{1+\eta}_{bdd}(\mathcal{X}^\alpha, \mathcal{X})$ for some $\alpha \in [0, 1)$ and $\eta \in (0, 1]$. We are interested in the singular limiting behavior of solutions in the case of resonance when $\mu^{(\xi)} \to \bar{\mu}$ as $\xi \to 0^+$ where $\bar{\mu} \in \sigma(A) = \{\mu_n, n \in N\}$ and $\mu^{(\xi)} \notin \sigma(A)$ for any $0 < \xi \ll 1$. The existence of solutions and the asymptotic expansions of equations of Sobolev type have been widely investigated by Sviridyuk *et al.* in a general context in [10], [11] and references therein.

Denote $P : \mathcal{X} \to \text{Ker }(A - \bar{\mu})$ the projector onto the kernel of $(A - \bar{\mu})$ and put $Q := I - P$ and $\varepsilon := (\bar{\mu} - \mu^{(\xi)})/\bar{\mu}^2$. Let us define the Hilbert spaces $X = Q\mathcal{X}$ and $Y = P\mathcal{X}$. The operator $(A - \mu^{(\xi)})Q = (A - \bar{\mu} + \varepsilon\bar{\mu}^2)Q$ is continuously invertible in X and, moreover, for any $0 \le \varepsilon \ll 1$,

$$A_\varepsilon := [(A - \bar{\mu} + \varepsilon\bar{\mu}^2)Q]^{-1} A^2$$

is again a self-adjoint operator in the Hilbert space X. Taking the projections of a solution w, $u := Qw$ and $v = Pw$, the Sobolev equation (28) can be rewritten as a system of equations

$$\begin{aligned} u_t + A_\varepsilon u &= G_\varepsilon(u, v) \in X \\ \varepsilon v_t + v &= \bar{\mu}^{-2} P f(u + v) \in Y \end{aligned} \tag{29}$$

where $G_\varepsilon(u,v) := [(A - \bar{\mu} + \varepsilon\bar{\mu}^2)Q]^{-1} f(u+v)$. The operator A_ε need not be positive definite. But it is bounded from below and this is why one can translate both the operator A_ε and the right hand side of the first equation such that $\sigma(A_\varepsilon) > 0$ for any small $0 \le \varepsilon \ll 1$. Notice that $X^\alpha = [D(A_0^\alpha)] = Q\mathcal{X}^\alpha$. Thus $G_\varepsilon \in C^{1+\eta}_{bdd}(X^\alpha \times Y, X)$ and $G_\varepsilon \to G_0$ as $\varepsilon \to 0^+$. Further, $A_0^{-1} : X \to X$ is a compact operator as well and $A_0 A_\varepsilon^{-1} - I = O(\varepsilon)$ in $L(X,X)$ when $\varepsilon \to 0^+$.

Hence all the assumptions of Theorem 8 are fulfilled provided that the number $\bar{\mu} \gg 1$ is large and the spectrum $\sigma(A_0) = \{\lambda_n,\ \lambda_n = \mu_n^2/(\mu_n - \bar{\mu}),\ n \in N,\ \mu_n \ne \bar{\mu}\}$ has sufficiently large spectral gaps. More precisely, $\lambda_n^\alpha/(\lambda_{n+1} - \lambda_n) \ll 1$. If the eigenvalues μ_n have the asymptotic $\mu_n = cn^2 + O(1)$ the latter condition is satisfied iff $\alpha < 1/2$ and $n \in N$ is large enough. We remind ourselves that the spectrum of the differential operator $Au := -\Delta u$, $A : H^2 \cap H_0^1(\Omega) \subset L_2(\Omega) \to L_2(\Omega)$, $\Omega = (0,1)^N$, has the above property for $N = 1$. In the dimension $N = 2$, it is known (cf. Richards [7]) that the spectrum of A has arbitrarily large spectral gaps. This yields that the condition $\lambda_n^\alpha/(\lambda_{n+1} - \lambda_n) \ll 1$ is satisfied for some $n \in N$ and the fractional power exponent $\alpha \ll 1$ small enough.

Having assured the hypotheses of Theorem 8 we may conclude that the Sobolev equation (28) has a C^1 smooth finite dimensional inertial manifold $\mathcal{M}_\xi$ for any $0 \le \xi \ll 1$. Moreover, the semiflow generated by (28) is stable in the resonance in a sense that the corresponding vector fields on $\mathcal{M}_\xi$ for $\xi = 0$ and $0 < \xi \ll 1$ are C^1-close to each other.

References

[1] Chow S.-N., Lu K., *Invariant manifolds for flows in Banach spaces*, J. of Differential Equations **74** (1988), 285–317.

[2] Foias C., Sell G.R., Temam R., *Inertial manifolds for nonlinear evolutionary equations*, J. of Differential Equations **73** (1988), 309–353.

[3] Henry D., *Geometric theory of semilinear parabolic equations*, Lecture Notes in Math. **840**, Springer Verlag, 1981.

[4] Marion M., *Inertial manifolds associated to partly dissipative reaction – diffusion systems*, J. of Math. Anal. Appl. **143** (1989), 295–326.

[5] Miklavčič M., *A sharp condition for existence of an inertial manifold*, J. of Dynamics and Differential Equations **3** (1991), 437–457.

[6] Mora X., Solà-Morales J., *The singular limit dynamics of semilinear damped wave equation*, J. of Differential Equations **78** (1989), 262–307.

[7] Richards J., *On the gaps between numbers which are the sum of two squares*, Adv. Math. **46** (1982), 1–2.

[8] Ševčovič D., *Limiting behaviour of invariant manifolds for a system of singularly perturbed evolution equations*, Math. Methods in the Appl. Sci. **17** (1994), 643–666.

[9] ———, *Limiting behavior of global attractors for singularly perturbed beam equations with strong damping*, Comment. Math. Univ. Carolinae **32** (1991), 45–60.

[10] Sviridyuk G.A., *The Deborah number and a class of semilinear equations of Sobolev type* (English translation), Soviet. Math. Doklady **44** No.1 (1992), 297–301.

[11] Sviridyuk G.A., Sukacheva T.G., *Cauchy problem for a class of semilinear equations of Sobolev type* (English translation), Sibirskii Matem. Zhurnal **31** No.5 (1990), 120–127.

[12] Vanderbauwhede A., Van Gils V.A., *Center manifolds and contraction on a scale of Banach spaces*, J. of Funct. Analysis **72** (1987), 209–224.

Department of Mathematical Analysis, Comenius University, Mlynská dolina, 842 15 Bratislava, Slovak Republic

E-mail: sevcovic@fmph.uniba.sk

(*Received* June 24, 1994)