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A note on group algebras of p-primary abelian groups

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Abstract. Suppose p is a prime number and R is a commutative ring with unity of characteristic 0 in which p is not a unit. Assume that G and H are p-primary abelian groups such that the respective group algebras RG and RH are R-isomorphic. Under certain restrictions on the ideal structure of R, it is shown that G and H are isomorphic.

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Suppose R is a commutative ring with unity of characteristic 0. If p is a prime number, and if G and H are p-primary abelian groups, the question arises of whether an R-isomorphism of the group algebras RG and RH implies that G and H are isomorphic. It is known that if $1/p \in R$, then one cannot expect $RG \cong RH$ to imply $G \cong H$. For example, in [U] it is shown that if R is an integral domain with sufficiently many p^k-th roots of unity for various integers $k \geq 1$, then $1/p \in R$ implies that the isomorphism class of RG is completely determined by $|G|$. In this brief note, we investigate conditions on R which guarantee that $G \cong H$ whenever $RG \cong RH$. Therefore, we assume throughout that $1/p \notin R$.

Let $\mathrm{inv}(R)$ be the set of prime numbers that are units in R, and let $\mathrm{zd}(R)$ be the set of prime numbers that are zero divisors in R. The characteristic of R is denoted by $\mathrm{char}(R)$. Throughout the remainder of this paper, our standing hypotheses are that R is a commutative ring with unity, $\mathrm{char}(R) = 0$, p is a prime number such that $p \notin \mathrm{inv}(R)$, and G and H are p-primary abelian groups.

Our first result appears in [U], but for the sake of completeness we include its short proof below. Its proof requires a special case of the main result of [M]; that is, if R is an integral domain and $RG \cong RH$, then $G \cong H$.

Proposition 1 ([U]). *If the additive group of R is torsion-free, then $RG \cong RH$ implies that $G \cong H$.*

Proof: Since $p \notin \mathrm{inv}(R)$, there exists a minimal prime ideal P of R such that $p \notin \mathrm{inv}(R/P)$. Moreover, R torsion-free means that $\mathrm{zd}(R) = \phi$. We conclude that R/P is an integral domain with $\mathrm{char}(R/P) = 0$ and $(R/P)G \cong (R/P)H$. It follows from the result of [M] mentioned above that $G \cong H$. □

The following consequence of Proposition 1 provides a necessary ingredient for the proofs of the subsequent results.

Proposition 2. *If $p \notin \mathrm{zd}(R)$, then $RG \cong RH$ implies $G \cong H$.*

Proof: Let T be the torsion subgroup of the additive group of R. Note that T is a proper ideal of R. We first claim that $p \notin \mathrm{inv}(R/T)$. Indeed, if $p \in \mathrm{inv}(R/T)$, then $n(pr-1) = 0$ for some $r \in R$ and integer $n > 0$. Since $p \notin \mathrm{inv}(R) \cup \mathrm{zd}(R)$, we may assume that p and n are relatively prime. Select integers s and t such that $sn + tp = 1$. Then, $0 = sn(pr-1) = (1-tp)(pr-1) = p(r - trp + t) - 1$, contradicting $p \notin \mathrm{inv}(R)$. Thus, $p \notin \mathrm{inv}(R/T)$ as claimed.

If $c \geq 0$ is the characteristic of R/T, then $c \in T$ and there exists an integer $m > 0$ such that $mc = 0$. Therefore, $c = 0$. Consequently, R/T is a torsion-free ring of characteristic 0 and $p \notin \mathrm{inv}(R/T)$. Since $(R/T)G \cong (R/T)H$, an application of Proposition 1 completes the proof. □

As usual, $J(R)$ denote the Jacobson radical of R.

Proposition 3. *Suppose $p \in J(R)$. Then $RG \cong RH$ implies that $G \cong H$.*

Proof: In view of Proposition 2, it suffices to show that R has a homomorphic image S of characteristic 0 with $p \notin \mathrm{inv}(S) \cup \mathrm{zd}(S)$.

First note that if p were contained in every minimal prime ideal of R, we would have $p^k = 0$ for some $k \geq 1$, contradicting $\mathrm{char}(R) = 0$. Set

$$I = \bigcap \{P : P \text{ is a minimal prime ideal of } R \text{ with } p \notin P\}$$

and let T_p denote the p-torsion of the additive group of R. Observe that $I + T_p$ is a proper ideal of R since $p \notin I$. We claim that $S = R/(I + T_p)$ has the desired properties.

Select a maximal ideal M containing $I + T_p$ and note that $p \in J(R)$ implies $p \in M$. Consequently, $p \notin \mathrm{inv}(S)$ since R/M is a homomorphic image of S and $p \notin \mathrm{inv}(R/M)$. Set $c = \mathrm{char}(S)$. If $c \neq 0$, there exist integers c' and m, with c' relatively prime to p and $m \geq 0$, such that $c = c'p^m \in I + T_p$. Thus, $c'p^{m+k} \in I$ for some $k \geq 1$. We conclude that $c' \in I \subseteq M$, which is absurd since $p \in M$ and M is proper. Therefore, $\mathrm{char}(S) = c = 0$. Finally, if $pr \in I + T_p$ for some $r \in R$, it follows that $r \in I$ and $p \notin \mathrm{zd}(S)$. □

If R is quasi-local with unique maximal ideal M, then $p \in M = J(R)$. Therefore, from Proposition 3 we obtain

Corollary 4. *If R is quasi-local, then $RG \cong RH$ implies $G \cong H$.*

As an application of Corollary 4, we obtain the following

Proposition 5. *Suppose the ideal Rp of R generated by p contains no nonzero idempotents. Then $RG \cong RH$ implies $G \cong H$.*

Proof: Let T_p denote the p-torsion subgroup of the additive group R. We claim that $I = T_p + Rp$ is a proper ideal of R. If not, $r + sp = 1$ for some $r \in T_p$ and $s \in R$. Therefore, $sp^{k+1} = p^k$ for some integer $k \geq 1$ and it follows by induction that $s^n p^{k+n} = p^k$ for every integer $n \geq 1$. In particular, $s^k p^{2k} = p^k$ and $(s^k p^k)^2 = s^{2k} p^{2k} = s^k p^k$. Since $s^k p^k \in Rp$ is idempotent, $s^k p^k = 0$. Consequently, $0 = s^k p^k p^k = s^k p^{2k} = p^k$, contradicting $\mathrm{char}(R) = 0$. Therefore, I is proper as claimed.

Select a maximal ideal M containing I and consider the localization R_M. Clearly $p \notin \mathrm{inv}(R_M)$ since $p \in M$. Moreover, if $c = \mathrm{char}(R_M)$, then $dc = 0$ for some $d \in R \backslash M$. Thus $c \in M$. Since $p \in M$, we have $c = p^m$ for some $m \geq 1$ or $c = 0$. If $c = p^m$, then $dp^m = 0$ implies that $d \in T_p \subseteq M$, a contradiction. Therefore, $\mathrm{char}(R_M) = 0$. An application of Corollary 4 now yields the result, since $R_M G \cong R_M \otimes_R RG \cong R_M \otimes_R RH \cong R_M H$. □

We summarize what we have proved in our final result.

Theorem 6. *Suppose R is a commutative ring with unity such that* $\mathrm{char}(R) = 0$ *and assume p is a prime number such that $p \notin \mathrm{inv}(R)$. If G and H are abelian p-groups such that $RG \cong RH$ as R-algebras, then $G \cong H$ in each of the following cases.*

(1) *Rp contains no nonzero idempotents (in particular, if R is indecomposable).*
(2) *$p \in J(R)$ (in particular, if R is quasi-local).*
(3) *$p \notin \mathrm{zd}(R)$ (in particular, if R is torsion-free).*

In closing we make a few remarks which may shed some light on the possible importance of results such as Theorem 6. First of all, one would ideally like to dispense with all conditions on R except for $\mathrm{char}(R) = 0$ and (the necessary hypothesis) $p \notin \mathrm{inv}(R)$. We formulate this as

Conjecture I. *Suppose* $\mathrm{char}(R) = 0$*, $p \notin \mathrm{inv}(R)$, and G and H are abelian p-groups with $RG \cong RH$. Then $G \cong H$.*

Also, we mention the long-standing conjecture in the modular case. As a reference, the reader is directed to G. Karpilovsky's excellent book [K], which is a fundamental source for any investigator in this area. We formulate Conjecture B on page 174 of [K] as

Conjecture II. *Suppose F is a field of characteristic $p \neq 0$ and G and H are abelian p-groups with $FG \cong FH$. Then $G \cong H$.*

It is easily proven that Conjectures I and II are equivalent (see, for example, [U]). That is, either both are true or both are false (or perhaps, undecidable in ZFC).

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