An inequality for the coefficients of a cosine polynomial

Horst Alzer

Abstract. We prove: If

$$\frac{1}{2}+\sum_{k=1}^{n} a_k(n)\cos(kx)\geq 0 \quad \text{for all} \quad x\in[0,2\pi),$$

then

$$1-a_k(n)\geq \frac{1}{2}\frac{k^2}{n^2} \quad \text{for} \quad k=1,\dots,n.$$

The constant $1/2$ is the best possible.

Keywords: cosine polynomials, inequalities
Classification: 26D05

In order to determine the saturation classes of optimal and quasi-optimal sequences (for details we refer to [1]), R.A. DeVore [1] proved in 1970 the following interesting integral inequality involving trigonometric polynomials.

Proposition. *Let n and k be integers with $n\geq k\geq 1$. For all non-negative trigonometric polynomials T_n of degree $\leq n$ with $\frac{1}{\pi}\int_{-\pi}^{\pi}T_n(x)\,dx=1$ we have*

$$\int_{-\pi}^{\pi}(\sin\frac{kx}{2})^2T_n(x)\,dx\geq\frac{1}{256\pi}\frac{k^2}{n^2}\,.$$

If we only consider cosine polynomials of the form

$$T_n(x)=\frac{1}{2}+\sum_{k=1}^{n}a_k(n)\cos(kx), \tag{1}$$

then the Proposition states that the inequality

$$1-a_k(n)\geq c_1(k^2/n^2) \tag{2}$$

with $c_1=1/(128\pi^2)$ is valid for all non-negative functions T_n and for all $k\in\{1,\dots,n\}$.

In 1970 E.L. Stark [2] discovered a better bound for $1-a_k(n)$. He established that (2) holds with $c_2=\pi^2/36$. This result was improved by Stark in 1976. In [3] he proved the validity of (2) with $c_3=\pi/9$. In the same paper he mentioned that the "problem of determining the optimal constant ... remains open" [3, p. 71]. To the best of my knowledge no solution of this problem has been published until now. It is the aim of this note to show that the best possible constant is $c=1/2$.

Theorem. *For all non-negative cosine polynomials* (1) *we have*

$$1 - a_k(n) \geq \frac{1}{2}\frac{k^2}{n^2} \qquad (k = 1, \ldots, n). \tag{3}$$

The constant $1/2$ *is the best possible.*

PROOF: As in [3] we define

$$u = u_k(n) = \frac{\pi}{[n/k] + 2} \qquad (1 \leq k \leq n).$$

($[x]$ denotes the greatest integer $\leq x$.) Then we have

$$\frac{\pi}{3}\frac{k}{n} \leq u \leq \frac{\pi}{3}\,. \tag{4}$$

Since the function $x \mapsto (1 - \cos(x))/x^2$ is strictly decreasing on $(0, \pi]$, we obtain

$$\frac{1 - \cos(u)}{u^2} \geq \frac{1 - \cos(\pi/3)}{(\pi/3)^2}\,, \tag{5}$$

so that (5) and the left-hand inequality of (4) yield

$$1 - \cos(u) \geq 9u^2/(2\pi^2) \geq k^2/(2n^2). \tag{6}$$

From (6) and

$$|a_k(n)| \leq \cos(u) \tag{7}$$

we conclude

$$1 - a_k(n) \geq 1 - \cos(u) \geq k^2/(2n^2).$$

If we set $T_n(x) = \frac{1}{2} + \frac{1}{2}\cos(nx)$, then the sign of equality holds in (3) for $k = n$, so that the constant $1/2$ cannot be replaced by a greater number. □

Remarks. (1) The proof of the Theorem reveals that the inequality (3) is strict for $k = 1, \ldots, n - 1$.

(2) The "extremely important" [3, p. 71] inequality (7) is due to J. Egerváry and O. Szász; see [4]. Concerning different proofs and extensions of (7) we refer to [3] and the references therein.

References

[1] DeVore R.A., *Saturation of positive convolution operators*, J. Approx. Th. **3** (1970), 410–429.

[2] Stark E.L., *Über trigonometrische singuläre Faltungsintegrale mit Kernen endlicher Oszillation*, Dissertation, TH Aachen, 1970.

[3] ———, *Inequalities for trigonometric moments and for Fourier coefficients of positive cosine polynomials in approximation*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. **544–576** (1976), 63–76.

[4] Szegö G., *Koeffizientenabschätzungen bei ebenen und räumlichen harmonischen Entwicklungen*, Math. Annalen **96** (1926–27), 601–632.

Morsbacher Str. 10, 51545 Waldbröl, Germany

(*Received* November 2, 1994)