On essential sets of function algebras in terms of their orthogonal measures

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Abstract. In the present note, we characterize the essential set of a function algebra defined on a compact Hausdorff space X in terms of its orthogonal measures on X.

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Let X be a compact Hausdorff topological space. Denote by $C(X)$ the commutative Banach algebra, consisting of all continuous complex-valued functions on X (with respect to usual point-wise algebraic operations) endowed with the sup-norm.

By a *function algebra* on X we mean any closed subalgebra of $C(X)$ which contains constant functions on X and which separates points of X.

Definition. A function algebra A on X is said to be a *maximal* one if it is a proper subset (i.e., a proper subalgebra) of $C(X)$ and has the following property: whenever B is a function algebra on X, $B \supset A$, then either $B = A$ or $B = C(X)$.

A being a function algebra on X, the closed subset E is said to be an *essential set of* A if the following conditions are fulfilled:

(*) A consists of all continuous prolongations of functions in the algebra of restrictions A/E (i.e., the algebra of all restrictions of functions in A from the set X to its subset E).

(**) Whenever a closed subset F of X has the same property as E in (*), then $E \subset F$ (or, E is a unique minimal closed subset of X satisfying the condition (*)).

The notion "essential set" is due to Bear, who proved in [1] that any maximal algebra on X has an essential set.

Hoffman and Singer in [2] found an essential set of any, not necessarily maximal, function algebra on X.

Denote by $M(X)$ the space of all complex Borel regular measures on X, i.e., by the Riesz Representation Theorem, the dual space of $C(X)$.

The *annihilator* $A^\perp$ of a function algebra A is defined to be the set of all measures $m \in M(X)$ such that $\int f\,dm = 0$ for any $f \in A$, or the set of all measures *orthogonal* to A. The dual space A' of A is then canonically isomorphic to the quotient space $M(X)/A^\perp$.

Now endow $M(X)$ with the weak-star topology: it is well known that $M(X)$ becomes a locally convex topological linear space with the dual space $C(X)$.

Our aim here is to characterize the essential set of a function algebra A by means of the properties of the measures in $A^\perp$. Remark that our construction is rather simpler than the classical one.

Theorem. *Let A be a function algebra on X. Denote by E the closure of the union of all closed supports of measures in $A^\perp$. Then E is the essential set of A.*

PROOF: Let $f \in C(X)$, $g \in A$ and let $f/E = g/E$, where f/E denotes the restriction of the function f from X to E. If for $m \in A^\perp$ we denote $M = \operatorname{spt}(m)$, then

$$\int f\,dm = \int_M f\,dm = \int_M g\,dm = \int g\,dm = 0,$$

hence f is orthogonal to $A^\perp$ and, by Banach theorem, $f \in A$. It means that E has the property (*) from Definition.

Now let a closed subset K have the property (*); we shall prove that $K \supset E$. Suppose that $K \not\supset E$. Then there is a measure $m \in A^\perp$, whose closed support is not a subset of K. Take $x \in \operatorname{spt}(m) \setminus K$. Let V be an open neighbourhood of x in X such that its closure $\overline{V}$ is disjoint with K. We shall find a function $f \in C(\overline{V})$ which fulfills the following two conditions:

$$\operatorname{spt}(f) \subset V, \ \int_V f\,dm \neq 0,$$

where $\operatorname{spt}(f)$ means the closed support of f. Denote by g such a function in $C(X)$, which is equal to f on $\overline{V}$ and equal to 0 off $\overline{V}$. Then $g/K = 0 \in A/K$, but

$$\int g\,dm = \int_V g\,dm = \int_V f\,dm \neq 0$$

and then $g \not\perp A^\perp$, so $g \notin A$. It follows that K has not the property (*). $\square$

Now the following question arises: whether the word "closure" in Theorem may be omitted, or whether the essential set E of a function algebra A on X is composed of the union of closed supports of all measures in $A^\perp$, without closure. We shall show that it is true if X is a metric space (Proposition), but in general it is not the case (Example).

Proposition. *Let X be a compact metric space, A a function algebra on X. Then the essential set E of A is equal to the union of closed supports of all measures in $A^\perp$. (Especially, the union of closed supports of all orthogonal measures is a closed set.)*

PROOF: Let $x \in E$. We shall find the measure $m \in A^\perp$ such that $\operatorname{spt}(m) \ni x$. Denote by U_n, $n = 1, 2, \dots$, the open balls in X with centres at x and radii $\frac{1}{n}$. We shall construct a finite or infinite sequence of measures $m_n \in A^\perp$ such that

$$|m_n|(X) \le 1, \tag{1}$$

$$(\operatorname{spt}(m_n) \smallsetminus \bigcup_{k=1}^{n-1} \operatorname{spt}(m_k)) \cap U_n \overset{def}{=} M_n \neq \emptyset \text{ and then } |m_n|(M_n) > 0, \tag{2}$$

$$|m_n|(X) < \min_{1 \le k \le n-1} |m_k|(M_k), \tag{3}$$

where $|m|$ means a total variation of a measure m.

By the Theorem, we can find a measure $m_1 \in A^\perp$ such that $|m_1|(X) = 1$ for which $\operatorname{spt}(m_1) \cap U_1 \neq \emptyset$. If $x \in \operatorname{spt}(m_1)$, the proof is finished. If it is not the case, then, by the Theorem, there exists the measure $m_2 \in A^\perp$ such that $(\operatorname{spt}(m_2) \smallsetminus \operatorname{spt}(m_1)) \cap U_2 \neq \emptyset$; (2) follows. Multiplying m_2 by a small enough nonzero constant, we can reach fulfilling (1) and (3). If $x \in \operatorname{spt}(m_2)$, we are done. In the opposite case, we shall continue the construction ...

In the case the sequence $\{m_n\}$ is finite, the proof is finished. If it is not the case, put

$$m = \sum_{n=1}^{\infty} \frac{1}{2^n} m_n.$$

By (1), it is $m \in M(X)$. Also $m \in A^\perp$ because $m_n \perp A$.

Take an arbitrary n. By (2), it is $|m_n|(M_n) > 0$, while $|m_k|(M_n) = 0$ for $1 \le k \le n-1$. By (3), it is

$$|m|(M_n) = |\sum_{k=n}^{\infty} \frac{1}{2^k} m_k(M_n)| \ge \frac{1}{2^n}|m_n|(M_n) - \sum_{k=n+1}^{\infty} \frac{1}{2^k}|m_k|(X) \ge$$

$$\ge \frac{1}{2^n}|m_n|(M_n) - \sum_{k=n+1}^{\infty} \frac{1}{2^k}|m_k|(X) > \frac{1}{2^n}|m_n|(M_n) - \frac{1}{2^n}|m_n|(M_n) = 0$$

and then $\operatorname{spt}(m) \cap U_n \neq \emptyset$. Since n was arbitrary, Proposition follows. $\square$

Now, we shall construct a function algebra A on X such that there exists a point $x \in E$ which is not contained in the closed support of any measure in $A^\perp$.

Example. *Let us denote by ω_1 the first uncountable ordinal number, put*

$$\Omega = \{\omega \textit{ ordinal; } \omega \leq \omega_1\},$$

let C be the closed unit disk in the complex plane. Denote by Y the cartesian product $C \times \Omega$ and let X arises from Y by "collapsing" the "last disk" $C \times \{\omega_1\}$ into one point, say x_1, i.e., $X = Y/C \times \{\omega_1\}$. Let the algebra A consist of all functions f continuous on X such that, for a fixed ordinal ω, $\omega < \omega_1$, the function $z \mapsto f(z, \omega)$ is holomorphic in $|z| < 1$. Then the singleton $\{x_1\}$ does not meet the closed support of any measure from $A^\perp$, while the essential set E of A is whole X.

Proof: (1) Any function $f \in C(\Omega)$ is constant on a neighbourhood of ω_1.

Let us suppose that $f(\omega_1) = 0$. Put, for natural n,

$$U_n = \{\omega \textit{ ordinal}; \omega \leq \omega_1 \ , |f(\omega)| < \frac{1}{n}\},$$
$$\omega^n = \sup \ \{\Omega \setminus U_n\}, \ \omega^0 = \sup_n \omega^n.$$

It follows from the properties of ordinal numbers that $\omega^n < \omega_1$, so $\omega^0 < \omega_1$, and $f = 0$ identically on the "ordinal interval" $[\omega^0, \omega_1]$.

(2) Any function in $C(X)$ is constant on some neighbourhood of x_1: this follows from (1).

(3) If $m \in A^\perp$ then $\mathrm{spt}(m) \cap \{x_1\} = \emptyset$.
Let $m \in A^\perp$. Then the ordinal

$$\omega_2 = \sup\{\omega \textit{ ordinal}; \ \omega < \omega_1, (z, \omega) \in \mathrm{spt}(m) \text{ for some } \ z, \ z \in C\}$$

is less than ω_1. Now let $f \in A$ be a function which is equal to 0 on the set $S = (C \times [1, \omega_2])$ and equal to 1 on $X \setminus S$. If the measure m contains a nonzero multiple of the one-point mass at $\{x_1\}$, it does not annihilate f, a contradiction. It follows that $\mathrm{spt}(m) \subset S$.

(4) Any "non-collapsed" disk supports the measure $m \in M(X)$ for which

$$\int f \, dm = \int_0^1 \int_{C_r(0)} f(z) \, dz \, dr$$

where $C_r(0) = re^{it}$ for $t \in [0, 2\pi]$, $0 < r \leq 1$. But $\int f \, dm = 0$, by the classical Cauchy Integral Theorem, and $m \in A^\perp$. The union of such disks is $X \setminus \{x_1\}$. □

References

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