# Examples of discontinuous, divergence-free solutions to elliptic variational problems

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*Dedicated to Professor Jindřich Nečas on occasion of his sixty-fifth birthday*

*Abstract.* We give an example of a bounded discontinuous divergence-free solution of a linear elliptic system with measurable bounded coefficients in $\mathbb{R}^3$ and a corresponding example for a **Stokes-like system**.

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## 1. Introduction

In this note, we give a simple example of a discontinuous bounded divergence-free weak solution to a three-dimensional linear elliptic system with measurable bounded coefficients of the form

$$-D_\alpha(A_{ij}^{\alpha\beta}(x)D_\beta u^j) = 0, \qquad i = 1,2,3 \tag{1.1}$$

(the summation convention has been adopted throughout the paper). We will suppose that

$$A := (A_{ij}^{\alpha\beta}) \in L^\infty(B, \mathbb{R}^{81}), \qquad B = \{x \in \mathbb{R}^3 : |x| < 1\}, \tag{1.2}$$

$$u^j : B \to \mathbb{R}, \qquad j = 1,2,3 \tag{1.3}$$

and that there is a constant $\lambda > 0$ such that

$$A_{ij}^{\alpha\beta}(x)\xi_\alpha^i \xi_\beta^j \geq \lambda|\xi|^2 \tag{1.4}$$

for every $\xi \in \mathbb{R}^9$ and almost every $x \in B$.

By a (weak) solution to the system (1.1) we understand a function

$$u \in W_{\text{loc}}^{1,2}(B, \mathbb{R}^3) = \left\{u : \int_K |\nabla u|^2 dx < +\infty \quad \forall K \subset\subset B\right\} \tag{1.5}$$

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such that (1.1) holds in the sense of distributions.

On account of the famous example of De Giorgi [2],[5] it is well known that solutions of linear elliptic systems with coefficients in $L^\infty$ are not Hölder-continuous in general.

Generalizing De Giorgi's example J. Souček in [14] gave a construction of an elliptic system, which has a solution which is discontinuous on a dense countable set. This shows that "partial regularity" (see [5], [12]) does not hold for solutions of linear systems with coefficients, which are only in $L^\infty$. Using Souček's "algorithm", John, Malý and Stará in [7] constructed a linear elliptic system with $L^\infty$-coefficients and its bounded weak solution for any given $F_\sigma$-set $F$ in $\mathbb{R}^3$, which is essentially discontinuous on $F$ and essentially continuous on $\mathbb{R}^3 \setminus F$.

Another modification of De Giorgi's example can be found in Leonardi [10] (see also [13]): this example shows that generally solutions of linear elliptic systems of the type (1.1) have no better integrability properties than the Sobolev-embedding in question tells us (compare also [3] for a nonlinear example). On the other hand the author presents estimates for the gradient of the solution to a system of type (1.1) in Morrey-spaces and weighted Morrey-spaces which are close to the counterexamples and he improves in some sense Koshelev's condition number result (see [7], [8], [10], [13]).

Further regularity results in Morrey-spaces or $L^p$-spaces for solutions of elliptic systems with coefficients in $L^\infty \cap VMO$ (the space of functions with vanishing mean oscillation) are established in [15]. These results are in some sense "intermediate" between Campanato's result ([1]) for systems with continuous coefficients and the counterexamples because of the "embedding $C^0 \hookrightarrow L^\infty \cap VMO$" and the fact that the coefficients in De Giorgi's counterexample have the property that their gradient belongs to the space weak-$L^n$. As a consequence of Poincaré's inequality we have that $W^{1,n}$-functions have locally vanishing mean oscillation.

In this situation a regularity result of De Giorgi-Moser-Nash-type for solutions of linear elliptic systems with $L^\infty$ — coefficients can be expected only if one finds new additional structure conditions for such systems and/or their solutions. For example one can ask whether weak solutions to our elliptic system which satisfy in addition an "incompressibility condition"

$$\operatorname{div} u = D_i u^i = 0$$

are regular, or, more general, whether all solutions of linear elliptic systems with $L^\infty$-coefficients are regular, if the mean flux of the given boundary function through the boundary of the given domain is zero (see [13,14]). Our example shows that the answer to these two questions must be negative.

Another still open question (at least to the authors' knowledge) is, whether minima of "isotropic" quadratic functionals of the type

$$J(u) := \frac{1}{2}\int_B W(Du)\,dx = \frac{1}{2}\int_B A_{ij}^{\alpha\beta}(x) D_\beta u^j D_\alpha u^i\,dx, \quad u \in W^{1,2}(B, \mathbb{R}^3)$$

are regular (see again [13]).

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## 2. The examples

For our examples we use Souček's algorithm to construct for a given $W^{1,2}$-function an elliptic system to which it is the solution. To construct a divergence-free $W^{1,2}$-function in the unit ball $B$ we call to mind that such a function must be a curl because of Poincaré's Lemma for star-convex open sets (see for example [11]).

Our divergence-free function in $B$ is

$$\begin{aligned} u(x) :=& curl \begin{pmatrix} |x| \\ |x| \\ |x| \end{pmatrix} = \frac{1}{|x|} \begin{pmatrix} x_2 & - & x_3 \\ x_3 & - & x_1 \\ x_1 & - & x_2 \end{pmatrix} \\ =& \frac{1}{|x|} \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} =: \frac{1}{|x|} Tx \,, \end{aligned} \tag{2.1}$$

which is an element of $L^\infty \cap W^{1,2}(B, \mathbb{R}^3)$, but is not continuous at $x = 0$ (in fact we have $Du \in L^p(B, \mathbb{R}^{3\times 3})$ for every $p < 3$). Notice that the mean flux of $u$ through $\partial B$ is zero.

Defining

$$b^i_\alpha(x) := \frac{F}{|x|} \left\{ t^i_\alpha + \frac{x_\alpha t^i_l x_l}{|x|^2} \right\} \tag{2.2}$$

for some positive constant $F$ and the matrix $T = (t^i_\alpha)$ from (2.1), one verifies

$$b^i_\alpha \in L^p(B) \quad \forall\, p < 3, \tag{2.3}$$

and

$$D_\alpha b^i_\alpha = 0 \quad \text{for } \ x \neq 0 \,. \tag{2.4}$$

Setting

$$\begin{aligned} d^i_\alpha :=& b^i_\alpha - D_\alpha u^i \\ =& \frac{1}{|x|} \left\{ (F-1) t^i_\alpha + (F+1) \frac{x_\alpha t^i_l x_l}{|x|^2} \right\} \,, \end{aligned} \tag{2.5}$$

elementary calculations show that for $F > 1$

$$d^i_\alpha\, D_\alpha u^i \geq \frac{1}{|x|^2}(2F-2) > 0\,, \tag{2.6}$$

$$\frac{b^i_\alpha\, d^i_\alpha}{d^i_\alpha\, D_\alpha u^i} \leq \frac{18F^2 - 2F}{2F-2} =: M < +\infty. \tag{2.7}$$

Therefore we can apply Lemma 1 of Souček's paper [14] to conclude that our function $u$ is the solution of the elliptic system

$$\int_B A^{\alpha\beta}_{ij}(x) D_\beta u^j D_\alpha \varphi^i \, dx = 0 \qquad \forall\, \varphi \in C^\infty_0(B, \mathbb{R}^3), \tag{2.8}$$

where the coefficients are defined by

$$A^{\alpha\beta}_{ij}(x) := \delta_{ij}\delta^{\alpha\beta} + \frac{d^i_\alpha d^j_\beta}{D_\rho u^s d^s_\rho}, \qquad \alpha, \beta, i, j = 1, 2, 3 \tag{2.9}$$

($\delta_{ij}, \delta^{\alpha,\beta}, \delta^i_\alpha$ are always Kronecker symbols).

We remark that one can construct corresponding functions $u$ in higher dimensions using the calculus of differential forms (replacing the special cross product of $\mathbb{R}^3$).

Furthermore a slight modification of Souček's above quoted Lemma 1 enables us to construct for given functions $(u, p)$ an elliptic system of Stokes-type

$$\begin{aligned} -D_\alpha(A^{\alpha\beta}_{ij} D_\beta u^j) + D_i p = 0,\\ div\ u = 0, \end{aligned}$$

such that $(u, p)$ is the solution of this system (for further details on these systems see [4], [6], [9]).

**Lemma.** *Let $b = (b^i_\alpha)$, $\alpha, i = 1, \dots, n$, be a matrix of $L^2$-functions such that*

$$\int_B b^i_\alpha D_\alpha \varphi^i \, dx = 0 \qquad \forall\, \varphi \in C^\infty_0(B, \mathbb{R}^n)\ . \tag{2.10}$$

*Further assume that $p \in L^2 \cap W^{1,1}(B)$ and define the matrix $c$ as*

$$c^i_\alpha := b^i_\alpha + p\delta^i_\alpha\ , \tag{2.11}$$

*and the matrix $d$ as*

$$d^i_\alpha := c^i_\alpha - D_\alpha u^i, \quad u \in W^{1,2}(B, \mathbb{R}^n)\ . \tag{2.12}$$

*Moreover we suppose, that*

$$d^i_\alpha D_\alpha u^i > 0 \tag{2.13}$$

*and that there is a positive constant $M$, such that*

$$\frac{c^i_\alpha d^i_\alpha}{D_\alpha u^i d^i_\alpha} \leq M\,. \tag{2.14}$$

*If $u$ is divergence-free, it is a solution to the elliptic system of Stokes-type:*

$$\begin{aligned} &\int_B A^{\alpha\beta}_{ij} D_\beta u^j D_\alpha \varphi^i \, dx = 0 \qquad \forall\, \varphi \in C^\infty_0(B, \mathbb{R}^n), \;\; div\ \varphi = 0, \\ &\qquad\qquad div\ u = 0, \end{aligned} \tag{2.15}$$

*where the coefficients are given once more by* (2.9).

Proof: This is done by easy calculations. The ellipticity corresponds to condition (2.13) and the boundedness to (2.14). □

Using this lemma we get an example of a singular solution to a system of Stokes-type in $\mathbb{R}^3$. We take $u$ as in (2.1) and define $p$ by

$$p(x) := \frac{1}{|x|} \in L^r \cap W^{1,s}(B) \qquad \forall\, r < 3, s < \frac{3}{2}\,. \tag{2.17}$$

Defining $b$ as before, i.e.

$$b^i_\alpha := \frac{1}{|x|}\left\{ F t^i_\alpha + F \frac{x_\alpha t^i_l x_l}{|x|^2} \right\} \tag{2.18}$$

we get

$$c^i_\alpha := \frac{1}{|x|}\left\{ F t^i_\alpha + F \frac{x_\alpha t^i_l x_l}{|x|^2} + \delta^i_\alpha \right\} \tag{2.19}$$

and

$$d^i_\alpha := \frac{1}{|x|}\left\{ (F-1) t^i_\alpha + (F+1) \frac{x_\alpha t^i_l x_l}{|x|^2} + \delta^i_\alpha \right\}\,. \tag{2.20}$$

Using these definitions and observing $t^i_\alpha \delta^i_\alpha = 0$ and $x_i t^i_l x_l = 0$ we calculate

$$\begin{aligned} d^i_\alpha D_\alpha u^i &= \frac{1}{|x|^2}\left\{ 6(F-1) - (F-1)\frac{(t^i_l x_l)^2}{|x|^2} \right\} \\ &\geq \frac{1}{|x|^2}\,\{2F-2\} > 0\,, \end{aligned} \tag{2.21}$$

provided $F > 1$, and

$$
\begin{aligned}
b^i_\alpha d^i_\alpha &= \frac{1}{|x|^2}\left\{6F(F-1) + [F(F-1) + 2F(F+1)]\frac{(t^i_l x_l)^2}{|x|^2} + 3\right\} \\
&\le \frac{1}{|x|^2}\,\{18F^2 - 2F + 3\}.
\end{aligned} \tag{2.22}
$$

So we finally arrive at

$$
\frac{b^i_\alpha d^i_\alpha}{d^i_\alpha D_\alpha u^i} \le \frac{18F^2 - 2F + 3}{2F - 2} =: M < +\infty \tag{2.23}
$$

and in view of the variant of Souček's lemma we have proved that

$$
(u, p) = \frac{1}{|x|}(Tx, 1) \tag{2.24}
$$

is a singular solution to a system of Stokes-type, where $A^{\alpha\beta}_{ij}$ are defined as in (2.9).

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