$\bigcap$-compact modules

Tomáš Kepka

Abstract. The duals of $\cup$-compact modules are briefly discussed.

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In the following, R is a non-zero associative ring with unit and modules are unitary left R-modules.

It is well known and easy to see that the following conditions are equivalent for a module M:

(C1) If M_i, $i \in \omega$, is a countable family of submodules of M such that $\sum M_i = M$, then $\sum_{i \le n} M_i = M$ for some $n \in \omega$.

(C2) If $M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots$ is a chain of submodules of M such that $\bigcup M_i = M$, then $M_n = M$ for some $n \in \omega$.

(C3) If $\varphi : \coprod_\omega A_i \longrightarrow M$ is an epimorphism, then $\varphi(\coprod_{i \le n} A_i) = M$ for some $n \in \omega$.

(C4) If $\mu : M \longrightarrow \coprod_I A_i$ is a homomorphism, then there is a finite subset J of I such that $Im(\mu) \subseteq \coprod_J A_i$.

(C5) If $\mu : M \longrightarrow \coprod_\omega A_i$ is a homomorphism, then there is $n \in \omega$ such that $\mathrm{Im}(\mu) \subseteq \coprod_{i \le n} A_i$.

(C6) If Q is a cogenerator for $R-$Mod and if $\mu : M \longrightarrow Q^{(\omega)}$ is a homomorphism, then there is $n \in \omega$ such that $\mathrm{Im}(\mu) \subseteq Q^{(n)}$.

Such a module M will be called $\cup$-*compact* in this paper (other names: $\sum$-compact, $\coprod$-slender, dually slender, small, etc.). A proper subclass of $\cup$-compact modules is formed by modules M satisfying the following condition:

(C7) If N is a countably generated submodule of M, then there is a finitely generated submodule K of M such that $N \subseteq K$.

These modules will be called *strongly* $\cup$-*compact* (other names: $(\aleph_0, \aleph_0)$-reducing, countably finite, etc.).

Now, consider the duals of the above conditions:

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(D1) If $M_i, i \in \omega$, is a countable family of submodules of M such that $\bigcap M_i = 0$, then $\bigcap_{i \le n} M_i = 0$ for some $n \in \omega$.

(D2) If $M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots$ is a chain of submodules of M such that $\bigcap M_i = 0$, then $M_n = 0$ for some $N \in \omega$.

(D3) If $\varphi : M \longrightarrow \prod_\omega A_i$ is a monomorphism, then $\varphi^{-1}(\prod_{i \ge n} A_i) = 0$ for some $n \in \omega$.

(D4) If $\mu : \prod_I A_i \longrightarrow M$ is a homomorphism, then there is a cofinite subset J of K such that $\prod_J A_i \subseteq \mathrm{Ker}(\mu)$.

(D5) If $\mu : \prod_\omega A_i \longrightarrow M$ is a homomorphism, then there is $n \in \omega$ such that $\prod_{i \ge n} A_i \subseteq \mathrm{Ker}(\mu)$.

(D6) If $\mu : R^\omega \longrightarrow M$ is a homomorphism, then there is $n \in \omega$ such that $R^{(\omega - n)} \subseteq \mathrm{Ker}(\mu)$.

Clearly, the conditions (D1), (D2) and (D3) are equivalent (the corresponding modules will be called $\cap$-*compact*), the conditions (D5) and (D6) are equivalent (the corresponding modules are just the well known slender modules — see [1, Chapter III]), (D4) implies (D5) and modules satisfying (D4) form a subclass of slender modules. In contrast to the dual situation, the classes of $\cap$-compact and slender modules never coincide:

Proposition 1. (i) *There exist finitely cogenerated* (*and hence* $\cap$-*compact*) *modules which are not slender.*

(ii) *If* $M \neq 0$ *is a slender module, then* $M^{(\omega)}$ *is slender but not* $\cap$-*compact.*

PROOF: (i) No non-zero factormodule of $R^\omega / R^{(\omega)}$ is slender but some of these factors are finitely cogenerated.

(ii) Slender modules are closed under direct sums (see [3]). □

The next proposition collects several easy observations on $\cap$-compact modules:

Proposition 2. (i) *The class of* $\cap$-*compact modules is closed under isomorphic images, submodules, extensions and finite direct sums.*

(ii) *If* $A_i, i \in I$, *is an infinite family of non-zero modules, then neither* $\coprod A_i$ *nor* $\prod A_i$ *is* $\cap$-*compact.*

(iii) *The following are equivalent for a module* M*:*

1. M *is artinian.*
2. *Every factor of* M *is finitely cogenerated.*
3. *Every factor of* M *is* $\cap$-*compact.*

(iv) *Every finitely cogenerated module is* $\cap$-*compact.*

(v) *Every countably cogenerated* $\cap$-*compact module is finitely cogenerated.*

(vi) *If* N *is an essential submodule of* M *and* N *is* $\cap$-*compact, then* M *is* $\cap$-*compact.*

An interesting class of rings is that of (left) steady rings — see [2]. Of course, we shall define the dual: The ring R is said to be (*left*) *dually steady* if every $\cap$-compact module is finitely cogenerated.

Lemma 1. *The following conditions are equivalent:*

(i) *Every $\cap$-compact cyclic module is finitely cogenerated.*

(ii) *Every non-zero $\cap$-compact (cyclic) module has a non zero socle.*

(iii) *Every $\cap$-compact injective module is finitely cogenerated.*

(iv) *R is dually steady.*

PROOF: (ii) implies (iv). Let M be $\cap$-compact. By (ii), $S = \mathrm{Soc}(M)$ is essential in M. But S is also $\cap$-compact, and hence S is finitely generated and it follows that M is finitely cogenerated. $\square$

Left noetherian rings, left perfect rings and left semiartinian rings of countable Soc-length are known to be steady. As concerns the dual case, the following result is available:

Proposition 3. *R is dually steady in each of the following cases:*

(1) *R possesses only countably many left ideals I such that ${}_RR/I$ is cocyclic.*

(2) *R is a countable ring.*

(3) *R is right noetherian and every left ideal is a (two-sided) ideal.*

(4) *R is commutative noetherian.*

(5) *R is left semiartinian.*

(6) *For every non-zero left ideal I, the cyclic module ${}_RR/I$ is artinian.*

PROOF: (i) If (1) is true, then every cyclic module is countably cogenerated and the result follows by combination of Proposition 2 (v) and Lemma 1.

(ii) In this case, every cyclic module is countably cogenerated.

(iii) Suppose, on the contrary, that (3) is satisfied and R is not (left) dually steady. Denote by $\mathcal{M}$ the set of proper (left) ideals I such that the cyclic module ${}_RR/I$ is $\cap$-compact and with zero socle. According to Lemma 1, $\mathcal{M}$ is non-empty, and so let $K \in \mathcal{M}$ be a maximal element of $\mathcal{M}$.

Now, let $r \in R-K$ and $M = R/(K:r)_l$. Then $M \cong (Rr+K)/K \subseteq {}_RR/K$ and consequently M is $\cap$-compact and $\mathrm{Soc}(M) = 0$. On the other hand, $K \subseteq (K:r)_l$, and hence $K = (K:r)_l$. We have proved that K is a prime ideal.

Since $\mathrm{Soc}(R/K) = 0$, K is not a maximal ideal and $R \neq K + Rr$ for some $r \in R-K$. Put $K_i = K+Rr^i$ for every $i \geq 0$. Then $R = K_0 \supseteq K_1 \supseteq K_2 \supseteq \dots$ and $K_i \neq K$. Since R/K is $\cap$-compact, we can take $s \in \bigcap K_i - K$. Then $s = a_i + r_i r^i$ for some $a_i \in K$, $r_i \in R$ and we have $a_i - a_{i+1} = (r_{i+1}r - r_i)r^i \in K$ and $b_i = r_{i+1}r - r_i \in K$. Thus $r_i \in K + r_{i+1}R$, $K + r_0R \subseteq K + r_1R \subseteq K + r_2R \subseteq \dots$ and there is $n \geq 0$ such that $K + r_nR = K + r_{n+1}R$. Now, $r_{n+1} = a + r_nb$, $a \in K$, $b \in R$, and $r_n = r_{n+1}r - b_n = ar + r_nbr - b_n$, $r_n(1-br) = ar - b_n \in K$. But $1 - br \notin K$, and therefore $r_n \in K$ and $s = a_n + r_nr^n \in K$, a contradiction.

(iv) This case follows immediately from the preceding one.

(v) This case follows immediately from Lemma 1.

(vi) If $\mathrm{Soc}_l(R) \neq 0$, then Lemma 1 applies. Assume $\mathrm{Soc}_l(R) = 0$. Then R is not left artinian and there is a sequence $I_0 \supseteq I_1 \supseteq I_2 \supseteq \dots$ of left ideals such that $I_n \neq I = \bigcap I_i$ for every $n \geq 0$. According to (6), $I = 0$ and it implies that ${}_RR$ is not $\cap$-compact. Now, R is dually steady by Lemma 1 again. □

The following observation will help us to construct an example of a non-dually-steady ring:

Observation 1. Let R be an integral domain with a quotient field $Q \neq R$. The following conditions are equivalent:

(1) R is $\cap$-compact.
(2) ${}_RQ$ is strongly $\cup$-compact.

Moreover, if R is a valuation domain, then these conditions are equivalent to:

(3) ${}_RQ$ is $\cup$-compact.
(4) ${}_RQ$ is not countably generated.

Example 1. Let $G(+) = \mathbb{Z}(+)^{(\omega_1)}$ and let H be the set of $a \in G$ such that either $a = 0$ or $a \neq 0$ and $a(\alpha) > 0$, where $\alpha = \max(\mathrm{supp}(a))$. Then $H(+)$ is a subsemigroup of $G(+)$ and we denote by S the corresponding semigroup-ring $\mathbb{Z}_2[H]$. Further, denote by P the set of $x \in S$ such that $a_i \neq 0_H$, where $x = r_0a_0 + \dots + r_na_n$, $r_i \in \mathbb{Z}_2$, $a_i \in H$. Then P is a prime ideal of S and we finally put $R = S(S-P)^{-1} \subseteq Q$, Q being a quotient field of S. It is easy to check that R is a valuation domain and R is $\cap$-compact. Consequently, R is not dually steady. In view of Observation 1, R is not steady either.

Remark 1. It would be of some interest to know other examples of dually steady and non-dually-steady rings, especially from the following classes of rings: left noetherian rings, left perfect rings, (von Neumann) regular rings, left V-rings (or, more generally, left conoetherian rings). In this respect, it would be also nice to obtain some information on rings without non-zero slender modules (see Proposition 1). Among such rings we shall certainly find many left semiartinian rings, all right perfect rings and all complete principal ideal domains.

References

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Department of Algebra, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 00 Praha 8, Czech Republic

E-mail: kepka@karlin.mff.cuni.cz

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