Which topological spaces have a weak reflection in compact spaces?

Martin Maria Kovár

Abstract. The problem, whether every topological space has a weak compact reflection, was answered by M. Hušek in the negative. Assuming normality, M. Hušek fully characterized the spaces having a weak reflection in compact spaces as the spaces with the finite Wallman remainder. In this paper we prove that the assumption of normality may be omitted. On the other hand, we show that some covering properties kill the weak reflectivity of a noncompact topological space in compact spaces.

Keywords: weak reflection, Wallman compactification, filter (base), net, θ-regularity, weak $[\omega_1, \infty)^r$-refinability

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1. Preliminaries and introduction

Let X be a topological space. Recall that the Wallman compactification of X is defined as the set $\omega X = X \cup \{y \mid y$ is a nonconvergent ultra-closed filter in $X\}$, where 'ultra-closed' means maximal among all filters, having a base consisting of closed sets. The sets $\mathcal{S}(U) = U \cup \{y \mid y \in \omega X \setminus X, U \in y\}$, where U is open in X, constitute an open base of ωX (see [3]). Recall that every point of the remainder $\omega X \setminus X$ is closed in ωX; hence $\omega X \setminus X$ is a T_1-space.

Any compactification γX of X is said to be a *a weak reflection of* X in the class of compact spaces if for every compact Y and every continuous mapping $f : X \to Y$ there exists a mapping $g : \gamma X \to Y$ continuously extending f.

The notion of weak reflection is a natural generalization of the concept of *reflection.* It is well-known that any continuous mapping $f : X \to Y$ to a compact Hausdorff space Y may be uniquely factorized through the Čech-Stone compactification βcX of the completely regular T_1-modification cX of X. Then the space βcX is the reflection of X in compact Hausdorff spaces. Note that for a compact Hausdorff space Y, the Wallman compactification ωX also has the extension property described above, but ωX is not necessarily Hausdorff.

It is natural to ask whether every topological space has, at least, a weak reflection in general compact spaces. This question was asked by J. Adámek and J. Rosický [1] and was answered in the negative by M. Hušek [4]. In fact, he described some spaces having a weak reflection in compact spaces (Theorem H.1) and some spaces having no weak reflection in compact spaces (Theorem H.2). He also fully characterized all normal spaces which have a weak reflection in compact spaces; they are exactly the spaces with the finite Wallman (or, equivalently, Čech-Stone) remainder. In the first part of this paper we show that the assumption of normality may be omitted. In the second part we prove that the presence of some covering properties (close to paracompactness) imply that such spaces, if they are noncompact, have no weak reflection in compact spaces. Note that the covering properties, including compactness and paracompactness, are regarded without any separation axiom. A topological space is (*countably*) *compact* if every (countable) open cover of the space has a finite subcover. A topological space is said to be *paracompact* if its every open cover has an open locally finite refinement. Rimcompactness is also regarded without the separation axioms. A topological space is called *rimcompact*, if every point of the space has a local base of open sets with compact boundary.

The following Hušek's theorems will be our starting point. For the proofs, the reader is referred to [4].

Theorem H.1. *If the Wallman remainder of a topological space X is finite, then the Wallman compactification of X is the weak reflection of X in compact spaces.*

Theorem H.2. *If a topological space X contains an infinite family $\{X_n\}_{n\in\mathbb{N}}$ of closed noncompact subsets such that $X_n \cap X_m$ is compact for $n \neq m$, then X has no weak reflection in compact spaces.*

2. Results

Before we prove our main theorem, we need the following three lemmas.

Lemma 1. *Let X be an infinite topological* T_1*-space. Then X contains an infinite subspace with the discrete topology or an infinite subspace with the topology of finite complements.*

Proof: Suppose that X does not contain an infinite subspace having the topology of finite complements. Since X is infinite, we may assume, without loss of generality, that $\mathbb{N} = \{1, 2, \dots\} \subseteq X$. Evidently, every $M \subseteq \mathbb{N}$ with $\mathbb{N} \setminus M$ finite is open in $\mathbb{N}$. By induction we construct a discrete subspace of X:

(1). Let $N_1 = \mathbb{N}$. By the assumption, N_1 has not the topology of finite complements. Hence there exists $x_1 \in N_1$ such that x_1 has an open neighbourhood U_1 with $N_1 \setminus U_1$ infinite. We put $N_2 = N_1 \setminus U_1$.

(2_k). Suppose that there exist points $x_1, x_2, \dots, x_k \in \mathbb{N}$, open sets $U_1, U_2, \dots, U_k$ and infinite sets $N_1 = \mathbb{N}$, $N_{i+1} = N_i \setminus U_i$, $i = 1, 2, \dots, k$ such that the following conditions are satisfied:

(i)$_k$ $x_i \in N_i \cap U_i$ for $i = 1, 2, \dots, k$,

(ii)$_k$ if $k \geq 2$ then $\{x_1, x_2, \dots, x_{i-1}\} \cap U_i = \varnothing$ for $i = 2, 3, \dots, k$.

We shall prove (2_{k+1}).

Since N_{k+1} is infinite, it cannot have the topology of finite complements. Then there exists a point $x_{k+1} \in N_{k+1}$ and its open neighbourhood U_{k+1} with $N_{k+2} = N_{k+1} \setminus U_{k+1}$ infinite. It follows from (i)$_k$ and from the definition of the sets N_i

that $x_i \notin N_{i+1} \supseteq N_{k+1}$ for $i = 1, 2, \dots, k$. Hence $\{x_1, x_2, \dots, x_k\} \cap N_{k+1} = \varnothing$. The set $\{x_1, x_2, \dots, x_k\}$ is closed; therefore, without loss of generality, we may choose the set U_{k+1} such that $\{x_1, x_2, \dots, x_k\} \cap U_{k+1} = \varnothing$. That completes the induction.

Let $D = \{x_1, x_2, \dots\}$. We will show that D has the discrete topology. Indeed, for every $p = 2, 3, \dots$ it follows from $(\mathrm{ii})_p$ that $\{x_1, x_2, \dots, x_{p-1}\} \cap U_p = \varnothing$. For every $p, i = 1, 2, \dots$ by $(\mathrm{i})_{p+i}$ we obtain that $x_{p+i} \in N_{p+i} \subseteq N_{p+1} = N_p \setminus U_p$. It follows that $U_p \cap \{x_{p+1}, x_{p+2}, \dots\} = \varnothing$. Now, we have $U_p \cap D = \{x_p\}$, which completes the proof. □

The following lemma is contained in [4] as Corollary 1. We repeat it with the proof because of completeness.

Lemma 2. *Let X be a topological space. Suppose that $\omega X \setminus X$ contains an infinite subspace with discrete topology. Then there exists a sequence $H_1, H_2, \dots$ of closed noncompact subsets of X which are pairwise disjoint.*

Proof: Assume, without loss of generality, that $\mathbb{N} \subseteq \omega X \setminus X$ and the topology of $\mathbb{N}$, induced from ωX, is discrete. Let $\mathfrak{O}$ be the collection of all open sets in X. There exist open sets $U_n \in \mathfrak{O}$, $n \in \mathbb{N}$ such that $n \in \mathcal{S}(U_n)$ and $m \notin \mathcal{S}(U_n)$ for $n \neq m$. Since $U_n \in n$, there is some closed $G_n \in n$ with $G_n \subseteq U_n$. The sets $H_n = G_n \setminus \bigcup_{i=1}^{n-1} U_i$, where $n \in \mathbb{N}$, constitute the desired family.

Indeed, every H_n is closed and disjoint from H_m for $m \neq n$. The noncompactness of H_n follows from the fact that every $n \in \mathbb{N} \subseteq \omega X \setminus X$ constitutes a nonconvergent ultra-closed filter in X. □

To prove our main theorem, we need yet a lemma which is complementary to Lemma 2.

Lemma 3. *Let X be a topological space. Suppose that $\omega X \setminus X$ contains an infinite subspace having the topology of finite complements. Then there exists a sequence $H_1, H_2, \dots$ of closed noncompact subsets of X which are pairwise disjoint.*

Proof: We may assume, without loss of generality, that $\mathbb{N} \subseteq \omega X \setminus X$ and the topology of $\mathbb{N}$, induced from ωX, is the topology of finite complements. Denote by $\mathfrak{O}$ the collection of all open sets in X. By induction we define the desired sequence:

(1). Let $N_1 = \mathbb{N}$, $x_1 = 1$, $y_1 = 2$. There exists an open set $U_1 \in \mathfrak{O}$ such that $x_1 \notin \mathcal{S}(U_1)$ and $y_1 \in \mathcal{S}(U_1)$. Since $N_1 \cap \mathcal{S}(U_1) \neq \varnothing$, the set $N_1 \setminus \mathcal{S}(U_1)$ is finite and then the set $N_2 = N_1 \cap \mathcal{S}(U1)$ is infinite. Since $x_1 \notin \mathcal{S}(U_1)$ it follows that $U_1 \notin x_1$ which implies that $X \setminus U_1 \in x_1$. We put $H_1 = X \setminus U_1$. Evidently, H_1 is closed in X and nonempty since $H_1 \in x_1$. Moreover, it is noncompact because x_1 is a nonconvergent ultra-closed filter in X.

(2_k). Suppose that for some $k \geq 1$ there exist open sets $U_1, U_2, \dots, U_k \in \mathfrak{O}$, sets $N_1, N_2, \dots, N_{k+1} \subseteq \mathbb{N}$ and noncompact closed sets $H_1, H_2, \dots, H_k \subseteq X$ such that

(i)$_k$ $U_1 \supseteq U_2 \supseteq \cdots \supseteq U_k$,
(ii)$_k$ $N_{i+1} = N_i \cap \mathcal{S}(U_i)$ for $i = 1, 2, \ldots, k$,
(iii)$_k$ N_{k+1} is infinite,
(iv)$_k$ $H_i \subseteq (X \setminus U_i) \cap U_{i-1}$ for $i = 2, 3, \ldots, k$.

We shall prove that (2_{k+1}) is fulfilled.

By (iii)$_k$ there are two distinct points $x_{k+1}, y_{k+1} \in N_{k+1}$ and an open set $U_{k+1} \in \mathfrak{O}$ such that $x_{k+1} \notin \mathcal{S}(U_{k+1})$ a $y_{k+1} \in \mathcal{S}(U_{k+1})$. Because $N_{k+1} \subseteq \mathcal{S}(U_k)$ by (ii)$_k$, one can easily check that we may assume $U_{k+1} \subseteq U_k$. Hence (i)$_{k+1}$ is fulfilled. We put $N_{k+2} = N_{k+1} \cap \mathcal{S}(U_{k+1})$. The set $\mathbb{N} \cap \mathcal{S}(U_{k+1})$ is open in $\mathbb{N}$ and nonempty because it contains y_{k+1}. Hence its complement $\mathbb{N} \setminus \mathcal{S}(U_{k+1})$ is finite; therefore $N_{k+1} \setminus \mathcal{S}(U_{k+1})$ is also finite. It follows that N_{k+2} is infinite. Notice that (ii)$_{k+1}$ and (iii)$_{k+1}$ are satisfied. Since $x_{k+1} \notin \mathcal{S}(U_{k+1})$, it follows $U_{k+1} \notin x_{k+1}$ and then $X \setminus U_{k+1} \in x_{k+1}$. On the other hand, since $x_{k+1} \in N_{k+1} \subseteq \mathcal{S}(U_k)$, we have $U_k \in x_{k+1}$. Then there exists a set $G_{k+1} \in x_{k+1}$, closed in X, such that $G_{k+1} \subseteq U_k$. We put

$$H_{k+1} = G_{k+1} \cap (X \setminus U_{k+1}).$$

Evidently, H_{k+1} is closed in X and since $H_{k+1} \in x_{k+1}$ it is nonempty. Moreover, it is noncompact because x_{k+1} is a nonconvergent ultra-closed filter in X. Since $G_{k+1} \subseteq U_k$ it follows that $H_{k+1} \subseteq (X \setminus U_{k+1}) \cap U_k$. Hence (iv)$_{k+1}$ is fulfilled, which completes the induction.

Now, let $p, s \in \mathbb{N}$, $p < s$. Then, by (iv)$_s$ and (i)$_{s-1}$, it follows that $H_s \subseteq U_{s-1} \subseteq \cdots \subseteq U_p$. On the other hand, by (iv)$_p$ we have $H_p \subseteq X \setminus U_p$, which implies that $H_p \cap H_s = \varnothing$. It follows that $\{H_i\}_{i \in \mathbb{N}}$ is the desired sequence. $\square$

Theorem 1. *Let X be a topological space. The following statements are equivalent:*

(i) *The Wallman compactification of X is its weak reflection in compact spaces.*
(ii) *The space X has a weak reflection in compact spaces.*
(iii) *There exists $k \in \mathbb{N}$ such that any pairwise disjoint family of closed sets in X contains at most k noncompact elements.*
(iv) *Every infinite sequence $H_1, H_2, \ldots$ of closed sets such that $H_p \cap H_q$ is compact for $p \neq q$ has a compact member.*
(v) *Every infinite pairwise disjoint family of closed sets in X has a compact member.*
(vi) *The Wallman remainder of X is finite.*

Proof: We will show that (i) ⇒ (ii) ⇒ (iv) ⇒ (v) ⇒ (vi) ⇒ (i) and (vi) ⇒ (iii) ⇒ (v). But (i) ⇒ (ii), (iii) ⇒ (v) and (iv) ⇒ (v) are clear; Theorem H.2 implies that (ii) ⇒ (iv). The implication (vi) ⇒ (i) follows from Theorem H.1.

(v) ⇒ (vi): Suppose that $\omega X \setminus X$ is infinite. Since the Wallman remainder is always a T_1-space, it follows from Lemma 1 that $\omega X \setminus X$ contains an infinite discrete subspace or an infinite subspace with the topology of finite complements. Then, by Lemma 2 or Lemma 3, we obtain that there is a sequence $H_1, H_2, \dots$ of subsets of X which are closed, noncompact and pairwise disjoint.

(vi) $\Rightarrow$ (iii): Let $k \in \mathbb{N}$ be the cardinality of $\omega X \setminus X$. Assume that for $m \in \mathbb{N}$ there are pairwise disjoint, closed and noncompact sets $H_1, H_2, \dots, H_m \subseteq X$. Then, since every H_i is noncompact and closed, there are nonconvergent ultraclosed filters $y_1, y_2, \dots, y_m \in \omega X \setminus X$ such that $H_i \in y_i$ for every $i = 1, 2, \dots, m$. Since $H_p \cap H_q = \varnothing$ for $p \neq q$, it follows that $y_p \neq y_q$. Therefore $m \leq k$, which completes the proof. $\square$

If X is a normal T_1-space, then its Wallman and Čech-Stone compactifications coincide. Hence the following Hušek's result [4] has here its natural place:

Corollary 1 (Hušek). *A normal* T_1*-space has a weak reflection in compact spaces if and only if its Čech-Stone remainder is finite.*

In the next part we will show that there is a number of topological spaces which have no weak reflection in compact spaces. In fact, we may say that most 'nonpathological' spaces, satisfying some of higher covering properties (which will be specified later), are exactly the case. For example, any paracompact noncompact space has no weak reflection in compact spaces, which is an easy consequence of Hušek's Theorem H.2. Among others, we will generalize this observation in the following section.

Let X be a topological space. A filter base Φ in X has a *θ-cluster point* $x \in X$ if every closed neighbourhood H of x and every $F \in \Phi$ have a nonempty intersection. The filter base Φ *θ-converges* to its *θ-limit* x if for every closed neighbourhood H of x there is $F \in \Phi$ such that $F \subseteq H$. Evidently, any θ-limit of a filter base is its θ-cluster point. A net $\varphi(B, \geqslant)$ has a *θ-cluster point* (a *θ-limit*) $x \in X$ if x is a θ-cluster point (a θ-limit) of the derived filter base $\{\{\varphi(\alpha) \mid \alpha \geqslant \beta\} \mid \beta \in B\}$.

A topological space X is said to be *θ-regular* [5] if every filter base (or equivalently [6], every net) in X with a θ-cluster point has a cluster point. It is shown in [6] that the class of θ-regular spaces contains all regular, rimcompact and all paracompact spaces as well. Recall that the topological space X is said to be (*strongly*) *locally compact* [5] if every $x \in X$ has a compact (closed) neighbourhood. Evidently, a strongly locally compact space is θ-regular.

Corollary 2. *If X is a θ-regular space having a weak reflection in compact spaces, then X is strongly locally compact.*

Proof: We leave to the reader to check that if X is a θ-regular space, then every two points $x \in X$, $y \in \omega X \setminus X$ have in ωX open disjoint neighbourhoods (for more detail, see [7]). Let $\omega X \setminus X = \{y_1, y_2, \dots, y_n\}$ and let $x \in X$ be a fixed point; $U_1, U_2, \dots U_n$ be open neighbourhoods of $y_1, y_2, \dots, y_n$ such that $x \notin \mathrm{cl}_{\omega X} U_i$ for every $i = 1, 2, \dots, n$. We put $K = \omega X \setminus \bigcup_{i=1}^{n} U_i$. Then $K \subseteq X$ is the desired closed compact neighbourhood of x. $\square$

The following example illustrates the necessity of the assumption of θ-regularity in Corollary 2.

Example 1. There exists a topological space X with the weak compact reflection which is not locally compact.

Construction: Let $X = \mathbb{N} \cup \{0\}$. Let us define a topology on X by taking the sets $V(m) = \{k \cdot 2^m | k = 0, 1, \dots\}$ for $m = 0, 1, \dots$ and $U(a) = \{1, 2, \dots, a\}$ for $a = 1, 2, \dots$ as its subbase.

One can easily check that any two infinite closed sets in X have a nonempty intersection. Then there exists at most one nonconvergent ultra-closed filter in X. Hence $\omega X \setminus X$ is finite.

We will show that the point $0 \in X$ has no compact neighbourhood. Let H be a neighbourhood of 0. Then there is some $m = 1, 2, \dots$ with $0 \in V(m) \subseteq H$. The family $\Omega = \{V(m+1)\} \cup \{U(a) | a = 1, 2, \dots\}$ is a cover of X. Since the set $V(m) \setminus V(m+1) \subseteq H \setminus V(m+1)$ is infinite, no finite subfamily of Ω covers H. Observe that X is not locally compact, but it has the weak compact reflection.

□

Corollary 3. *A regular non-locally compact space has no weak reflection in compact spaces.*

Corollary 4. *A rimcompact non-locally compact space has no weak reflection in compact spaces.*

Lemma 4. *Let X be a topological space having a weak reflection in compact spaces. Then every sequence $\{x_n\}_{n\in\mathbb{N}}$ whose points are closed in X has a cluster point.*

Proof: Suppose that the sequence $\{x_n\}_{n\in\mathbb{N}}$ consists of points which are closed in X and has no cluster point in X. Let $\eta : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ be any one-to-one mapping between $\mathbb{N}\times\mathbb{N}$ and $\mathbb{N}$. Then the sets $H_k = \left\{x_{\eta(k,n)} |\, n = 1, 2, \dots\right\}$, where $k \in \mathbb{N}$, constitute an infinite sequence of pairwise disjoint, closed and noncompact sets. Now, Theorem 1 completes the proof. □

Definition 1. Let X be a topological space and $x, y \in X$ be two points. We say that y *absorbs* x if every open neighbourhood of y contains x. A set $Y \subseteq X$ is said to be an *absorbing set of* X if every $x \in X$ is absorbed by some $y \in Y$. An absorbing set of X is called *point-closed* if its every point is closed in X.

Lemma 5. *Let X be a topological space which is* T_0 *and θ-regular. Then X contains a point-closed absorbing set.*

Proof: For every $x, y \in X$ let $x \leqslant y$ (or, equivalently, $y \geqslant x$) if and only if y absorbs x. Clearly, the relation $\leqslant$ is transitive and reflexive. Since X is a T_0-space, $\leqslant$ is also antisymmetric; hence it is an order on X. Let $M \subseteq X$ be a nonempty set which is a chain with respect to $\leqslant$. Pick any fixed $x \in M$ and take a closed neighbourhood G of x. For any $y \in M$, $y \geqslant x$ it follows, since y absorbs x, that $y \notin X \setminus G$. Hence $y \in G$, which implies that the net $id_M(M, \geqslant)$ θ-converges to x. Since X is θ-regular, $id_M(M, \geqslant)$ has a cluster point, say $z \in X$.

Let U be an open neighbourhood of z and take $x \in M$. It follows that there is some $y \in M$, $y \geqslant x$ such that $y \in U$. But y absorbs x, hence $x \in U$ as well. It follows that z absorbs every point of M which means that z is an upper bound of M. By Zorn's Lemma, every $x \in X$ is absorbed by some element of X which is maximal with respect to the order $\leqslant$. Let $Y = \{y | \, y \in X, y \text{ is maximal with respect to } \leqslant\}$. Obviously, Y is an absorbing subset of X. Let $y \in Y$, $x \in X$, $x \neq y$. It is not possible for x to absorb y since y is maximal. Then there exists an open neighbourhood V of x such that $y \notin V$. Therefore the set $\{y\}$ is closed in X and then Y is point-closed. □

Corollary 5. *Let X be a topological space having a weak reflection in compact spaces. If X is θ-regular or* T_1*, then it is countably compact.*

PROOF: For a T_1-space, the assertion is an immediate consequence of Lemma 4. Let X be a θ-regular space having a weak reflection in compact spaces. It suffices to show that every sequence $\{x_n\}_{n\in\mathbb{N}}$ in X has a cluster point.

At first, assume that X is a T_0-space. By Lemma 5, X has a point-closed absorbing set, say $Y \subseteq X$. Then every x_n is absorbed by some $y_n \in Y$. It follows from Lemma 4 that the sequence $\{y_n\}_{n\in\mathbb{N}}$ has a cluster point, say $x \in X$. Now, since y_n absorbs x_n, one can easily check that x is a cluster point of $\{x_n\}_{n\in\mathbb{N}}$.

Finally, suppose that X is not T_0 in general. For any $x, y \in X$ we put $x \sim y$ if and only if the points x, y absorb each other. The relation $\sim$ is a relation of equivalence on X; in fact, equivalent points have the same open neighbourhoods. Choose from each class of equivalence, associated with $\sim$, exactly one point $z \in X$ and denote by Z the set of all such points z. Clearly, Z is an absorbing set of X which is a T_0-space in the induced topology. It is easy to show that Z has a weak reflection in compact spaces and also is θ-regular. It follows from the previous paragraph that every sequence in Z has a cluster point. If $\{x_n\}_{n\in\mathbb{N}}$ is any sequence in X, by an analogous trick as before we choose for every x_n some $z_n \in Z$ absorbing x_n. Then the cluster point of $\{z_n\}_{n\in\mathbb{N}}$, say $x \in Z \subseteq X$, is a cluster point of $\{x_n\}_{n\in\mathbb{N}}$. That completes the proof. □

Example 2. Let X be the space constructed in Example 1. Then X has the weak compact reflection, but it is not countably compact. Observe that the space X is neither T_1 nor θ-regular.

In the following, for any set S, the cardinality of S is denoted by $|S|$. Let Φ be a family of subsets of X, $x \in X$. We denote by $\mathrm{ord}(x, \Phi) = |\{F| \, F \in \Phi, x \in F\}|$. Recall that a topological space X is said to be *weakly* $[\omega_1, \infty)^r$*-refinable* if for any open cover Ω, of uncountable regular cardinality, there exists an open refinement which can be expressed as $\bigcup_{\alpha\in A} \Phi_\alpha$ where $|A| < |\Omega|$ and if $x \in X$ there is some $\alpha \in A$ such that $0 < \mathrm{ord}(x, \Phi_\alpha) < |\Omega|$. This property is due to J.M. Worrell and H.H. Wicke; they proved that a countably compact weakly $[\omega_1, \infty)^r$-refinable space is compact. For the proof we refer the reader to [2]. The proof needs no separation axiom and works for general topological spaces. Remark that the class of weakly $[\omega_1, \infty)^r$-refinable spaces contains the classes of paracompact spaces, metacompact spaces and also a number of their generalizations (para-Lindelöf, σ-para-Lindelöf, screenable, σ-metacompact, meta-Lindelöf, submeta-Lindelöf, submetacompact, weakly θ-refinable, weakly $\delta\theta$-refinable spaces; for more detail, see [2]). The following corollary shows that such spaces, if they are noncompact and satisfy some slight separation axioms, have no weak reflection in compact spaces.

Corollary 6. *Let X be a noncompact topological space which is θ-regular or T_1. If X is weakly $[\omega_1, \infty)^r$-refinable, then it has no weak reflection in compact spaces.*

Proof: In the light of the result of Worrell and Wicke, the assertion immediately follows from Corollary 5. □

Since any paracompact space is θ-regular [6] and weakly $[\omega_1, \infty)^r$-refinable, the formerly mentioned case of a noncompact paracompact space now also easily follows from Corollary 6.

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Department of Mathematics, Faculty of Electrical Engineering and Informatics, Technical University of Brno, Technická 8, 616 69 Brno, Czech Republic

E-mail: kovar@dmat.fee.vutbr.cz

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