Comment.Math.Univ.Carolin. 37,1 (1996)135–146

Booleanization of uniform frames

B. Banaschewski*, A. Pultr**

Abstract. Booleanization of frames or uniform frames, which is not functorial under the basic choice of morphisms, becomes functorial in the categories with weakly open homomorphisms or weakly open uniform homomorphisms. Then, the construction becomes a reflection. In the uniform case, moreover, it also has a left adjoint. In connection with this, certain dual equivalences concerning uniform spaces and uniform frames arise.

Keywords: Booleanization, uniform frame, uniform space, weakly open maps and homomorphisms

Classification: 06D10, 06E15, 18A40, 18B30, 54C10, 54E15

The familiar topological fact that the regular open subsets of a space form a complete Boolean algebra, extended by Glivenko [6] to arbitrary frames, gives rise to the Booleanization of a *uniform* frame L, obtained by equipping the Boolean frame of regular elements of L with the uniformity induced by that of L. It is then straightforward, if not trivial, that any complete uniform frame is the completion of its Booleanization, while any Boolean uniform frame is the Booleanization of its completion, by a basic result of Isbell [8]. This paper investigates the functorial aspects of this situation.

First, we extend a result of Banaschewski-Pultr [5] for mere frames to the case of uniform frames (Proposition 1): as in [5], Booleanization becomes a reflection for uniform homomorphisms which are *weakly open*, characterized among others by the condition that dense elements are mapped to dense elements. Next we establish, in crucial contrast to the situation of mere frames, that Booleanization has a left adjoint, provided by completion (Proposition 2), which then induces an equivalence between Boolean uniform frames with all uniform homomorphisms and complete uniform frames with weakly open uniform homomorphisms (Proposition 3). In addition, based on suitable naturally arising uniformities, this leads to equivalences between totally bounded Boolean uniform frames with all uniform homomorphisms and compact regular frames with weakly open homomorphisms (Proposition 4), and between Boolean frames with all homomorphisms and Gleason frames with all weakly open homomorphisms (Proposition 5).

---

*Support of the Natural Sciences and Engineering Research Council of Canada and of the Categorical Topology Research group at the University of Cape Town is gratefully acknowledged.
**Support of the Natural Sciences and Engineering Research Council of Canada and the Grant Agency of the Czech Republic under Grant 201/93/950 is gratefully acknowledged.

Finally, we deal with the relationship between uniform frames and uniform spaces. First, we obtain the general result that the category of complete uniform spaces and uniformly continuous maps is dually equivalent, by the obvious functors, to the category of those completely uniform frames which we call weakly spatial (Proposition 6). This, in turn, leads to a further dual equivalence in which the maps on the frame side are restricted to weakly open uniform homomorphisms (Proposition 7), and then to a dual equivalence between the category of complete uniform spaces with weakly open uniformly continuous maps and the category of Boolean uniform frames which are separated by their Cauchy spectrum (Proposition 8).

We note that another study of the Booleanization of frames with additional structure is carried out in Banaschewski-Pultr [3], dealing with the case of metric frames. In broad outline, the results there parallel those presented here but the distinctive features of the two areas make several details considerably different; in particular, neither can be construed as a special case of the other.

0. Background and definitions

Recall that a *frame* is a complete lattice L in which

$$a \wedge \bigvee S = \bigvee \{a \wedge t \mid t \in S\}, \quad \text{for all} \quad a \in L \quad \text{and} \quad S \subseteq L,$$

and a *frame homomorphism* is a map $h : L \to M$ between frames preserving all finitary meets, including the unit (= top) e, and arbitrary joins, including the zero (= bottom) 0.

Any complete Boolean algebra is a frame, and the frame homomorphisms between complete Boolean algebras are exactly the complete Boolean homomorphisms. On the other hand, any frame L determines a complete Boolean algebra L_{**}, consisting of the elements $a = a^{**}$ of L, where $(\ )^*$ is the pseudocomplementation in L, that is, $x^* = \bigvee\{y \in L \mid y \wedge x = 0\}$ (Glivenko [6]). Note that meets in L_{**} are the same as in L while the join of any $S \subseteq L_{**}$ in L_{**} is $(\bigvee S)^{**}$; in particular, the map $L \to L_{**}$ taking each a to a^{**} is a frame homomorphism. The latter is characterized as the essentially unique dense homomorphism $h : L \to M$ onto a Boolean frame M, dense meaning that $h(a) = 0$ implies $a = 0$ (Isbell [8]).

For general facts concerning frames we refer to Johnstone [9] or Vickers [13].

In a frame L, a *cover* is any subset whose join is the unit, and, for covers A and B of L, $A \leq B$ ("A refines B") means that, for each $a \in A$, there exist $b \geq a$ in B. Further, for any cover A of L and any $c \in L$, we put

$$Ac = \bigvee\{x \in A \mid x \wedge c \neq 0\}$$

and define $A \leq^* B$ ("A star-refines B") for covers A and B of L to mean that the cover $\{Ac \mid c \in A\}$ refines B. Finally, a *uniformity* on L is a filter $\mathfrak{U}$ of covers of L (in the sense of $\leq$) for which each member of $\mathfrak{U}$ is star-refined by some member of $\mathfrak{U}$ and each $a \in L$ is the join of all $x \in L$ such that $Cx \leq a$ for some $C \in \mathfrak{U}$.

In the following, $L, M, \ldots$ will be *uniform frames*, that is, frames equipped with a specified uniformity. The latter will be denoted $\mathfrak{U}L, \mathfrak{U}M, \ldots$. Further, we allow the notational confusion between $L, M, \ldots$ and their underlying frames. A *uniform (frame) homomorphism* is a map $h : L \to M$ between uniform frames which is a frame homomorphism and preserves uniform covers, that is, $h[A] \in \mathfrak{U}M$ for any $A \in \mathfrak{U}L$. **UniFrm** will then be the resulting category.

A uniform frame homomorphism $h : L \to M$ is called a *surjection* if it is an onto map and $\mathfrak{U}M$ is generated by the $h[C]$, $C \in \mathfrak{U}L$. Explicitly, the latter means that, for each $B \in \mathfrak{U}M$, there exist $C \in \mathfrak{U}L$ such that $h[C] \leq B$. A uniform frame L is called *complete* if any dense surjection $M \to L$ is an isomorphism. Any uniform frame L has a *completion*, that is, a dense surjection $\gamma_L : CL \to L$ with complete CL, unique up to isomorphism, providing the coreflection of **UniFrm** to its full subcategory **CUniFrm** of complete uniform frames (Isbell [8]; also Kříž [11], Banaschewski-Pultr [2]). In the following, C will be the functor determined by completion.

Given a *uniform* frame L, a mere frame N, and an onto frame homomorphism $h : L \to N$, the latter can be made into a uniform frame homomorphism by endowing N with the uniformity generated by the covers $h[C]$, $C \in \mathfrak{U}L$, consisting of all covers of N refined by some such $h[C]$. That this really works is a consequence of the obvious inequality

$$h[A]h(x) \leq h(Ax),$$

for any cover A of L and any $x \in L$. Of course, with this uniformity on N, h becomes a surjection of uniform frames.

In particular, we can use this observation to introduce the *Booleanization* of a uniform frame L, denoted $\beta_L : L \to \mathfrak{B}L$, where $\mathfrak{B}L$ is the uniform frame with underlying frame L_{**} and uniformity generated by the covers

$$\{x^{**} \mid x \in C\} \quad (C \in \mathfrak{U}L),$$

and β_L maps $x \in L$ to x^{**}. The subcategory of **UniFrm** that these $\mathfrak{B}L$ belong to is the full subcategory $\mathfrak{B}$**UniFrm** determined by all Boolean uniform frames.

Now, if L is a complete uniform frame then $\beta_L : L \to \mathfrak{B}L$, as a dense surjection, is the completion of $\mathfrak{B}L$, by the uniqueness of completions. On the other hand, if M is a Boolean uniform frame then $\gamma_M : CM \to M$, as a dense homomorphism onto a Boolean frame, is the Booleanization of CM, by the uniqueness result of Isbell [8] quoted earlier. Hence, the correspondence between the objects of $\mathfrak{B}$**UniFrm** and **CUniFrm** given by completion and Booleanization are inverse to each other, up to isomorphism. In the following, the functorial aspects of this situation are studied.

1. Booleanization as reflection

A natural question concerning Booleanization of uniform frames is: which uniform homomorphisms make the correspondence $L \mapsto \mathfrak{B}L$ functorial so that the

Booleanization maps $\beta_L : L \to \mathfrak{B}L$ constitute a natural transformation? Below, we provide some characterizations where $h : L \to M$ is any map of uniform frames.

Lemma 1. *The following are equivalent:*

(1) *There exists* $\overline{h} : \mathfrak{B}L \to \mathfrak{B}M$ *such that* $\overline{h}\beta_L = \beta_M h$.
(2) *For any dense* $a \in L$, $h(0)$ *is dense.*
(3) *For any* $a \in L$, $h(a^{**}) \leq h(a)^{**}$.

Proof: For mere frames this is part of Theorem 4.3 in Banaschewski-Pultr [3], and hence it only remains to check that (3) $\Rightarrow$ (1) still holds in the uniform case. Now, $\mathfrak{U}(\mathfrak{B}L)$ is generated by the covers $\beta_L[C]$, $C \in \mathfrak{U}L$, and $\overline{h}[\beta_L[C]] = \beta_M[h[C]]$ belongs to $\mathfrak{U}(\mathfrak{B}M)$, by the definition of the latter and since h is uniform. Thus, $\overline{h}$ is uniform, as claimed. □

Following the terminology of [4], we shall call the $h : L \to M$ which satisfy the equivalent conditions of Lemma 1 *weakly open* and let $\mathbf{UniFrm}_{wo}$ then be the category of all uniform frames and their weakly open uniform homomorphisms. Note that, for Boolean L, any $h : L \to M$ is trivially weakly open since e is the only dense $a \in L$; in particular, $\mathfrak{B}\mathbf{UniFrm}$ is a subcategory of $\mathbf{UniFrm}_{wo}$.

For any weakly open $h : L \to M$, the $\overline{h} : \mathfrak{B}L \to \mathfrak{B}M$ of Lemma 1 is uniquely determined; we shall express this by putting $\mathfrak{B}h = \overline{h}$, obtaining a functor $\mathfrak{B} : \mathbf{UniFrm}_{wo} \to \mathfrak{B}\mathbf{UniFrm}$. Moreover, as an immediate consequence of Lemma 1, together with the obvious fact that $\beta_L : L \to \mathfrak{B}L$ is the identity map for Boolean L and weakly open for any L, we have the following uniform counterpart of Theorem 2.2 of Banaschewski-Pultr [5]:

Proposition 1. $\mathfrak{B}\mathbf{UniFrm}$ *is reflective in* $\mathbf{UniFrm}_{wo}$, *with reflection functor* $\mathfrak{B}$ *and reflection maps* $\beta L : L \to \mathfrak{B}L$.

Remark. For spatial frames, Lemma 1 is essentially due to Johnstone [10], using the alternative condition $h(a^*)^* = h(a)^{**}$ instead of (2) or (3), which is known to be equivalent to the latter by Banaschewski-Pultr [4]. As to terminology, following that of Johnstone [10] the homomorphisms in (1) would be called *skeletal*, which is derived from topological usage due to Mioduszewski-Rudolf [12]. On the other hand, somewhat earlier, Herrlich and Strecker [7] had introduced the term *demi-open* for continuous maps of this kind. We find it more suggestive to call these homomorphisms weakly open, especially in view of the analysis carried out in [4].

2. Booleanization as right adjoint

As the preceding section shows, it is quite straightforward that the functor $\mathfrak{B} : \mathbf{UniFrm}_{wo} \to \mathfrak{B}\mathbf{UniFrm}$ is left adjoint to the corresponding inclusion functor, once the rôle of the weakly open homomorphisms in this context has been recognized. We now turn to the rather more subtle result that $\mathfrak{B}$ also *has a left adjoint*, supplied by completion, which in turn leads to various noteworthy consequences.

We begin with a couple of lemmas.

Lemma 2. *For any uniform $h : L \to M$, Ch is an isomorphism iff h is a dense surjection.*

Proof: ($\Rightarrow$) If Ch is an isomorphism then $h\gamma_L = \gamma_M Ch$ is a dense surjection, and this makes h a dense surjection.

($\Leftarrow$) is proved in Banaschewski-Pultr [2]. $\square$

Lemma 3. *C induces a functor on* $\mathbf{UniFrm}_{wo}$.

Proof: For any $h : L \to M$ in **UniFrm**, we have the commuting square

$$\begin{array}{ccc} CL & \xrightarrow{\gamma_L} & L \\ {\scriptstyle Ch}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ CM & \xrightarrow[\gamma_M]{} & M \end{array}$$

with dense onto γ_L and γ_M. Now, if h is weakly open then, for any dense $a \in CL$, $\gamma_M Ch(a) = h\gamma_L(a)$, and hence $Ch(a)$ is dense since dense onto homomorphisms obviously preserve and reflect denseness of elements. $\square$

Now we come to the desired main result.

Proposition 2. $\mathfrak{B}$: $\mathbf{UniFrm}_{wo} \to \mathfrak{B}\mathbf{UniFrm}$ *has a left adjoint, given by completion.*

Proof: For any Boolean uniform frame M, both

$$\gamma_M : CM \to M \quad \text{and} \quad \beta_{CM} : CM \to \mathfrak{B}CM$$

are dense homomorphisms onto a Boolean frame, and as noted earlier there exists an isomorphism $\mu_M : M \to \mathfrak{B}CM$, necessarily unique, such that $\mu_M\gamma_M = \beta_{CM}$.

Further, for any uniform frame L, we have the commuting square

$$\begin{array}{ccc} CL & \xrightarrow{C\beta_L} & C\mathfrak{B}L \\ {\scriptstyle \gamma_L}\big\downarrow & & \big\downarrow{\scriptstyle \gamma_{\mathfrak{B}L}} \\ L & \xrightarrow[\beta_L]{} & \mathfrak{B}L \end{array}$$

where $C\beta_L$ is an isomorphism by Lemma 2. Hence we have $\rho_L : C\mathfrak{B}L \to L$, given by $\rho_L = \gamma_l(C\beta_L)^{-1}$, weakly open since γ_L is.

It is straightforward to show that these maps define natural transformations $\mu : \mathrm{Id} \to \mathfrak{B}C$ and $\rho : C\mathfrak{B} \to \mathrm{Id}$. We claim they satisfy the identities that make C left adjoint to $\mathfrak{B}$.

For the identity

$$\rho_{CM} \circ C\mu_M = \mathrm{id}_{CM}$$

note that

$$\gamma_M \circ \rho_{CM} \circ C\mu_M \circ C\gamma_M = \gamma_M \circ \gamma_{CM} \circ (C\beta_{CM})^{-1} \circ C\beta_{CM} = \gamma_M \circ \gamma_{CM}$$

while the commuting square

$$\begin{tikzcd}
CCM \arrow[r, "C\gamma_M"] \arrow[d, "\gamma_{CM}"'] & M \arrow[d, "\gamma_M"] \\
CM \arrow[r, "\gamma_M"'] & M
\end{tikzcd}$$

shows that $\gamma_M \circ \gamma_{CM} = \gamma_M \circ C\gamma_M$. Hence

$$\gamma_M \circ \rho_{CM} \circ C\mu_M \circ C\gamma_M = \gamma_M \circ C\gamma_M$$

where both $C\gamma_M$ and γ_M can be cancelled, the latter by denseness and the regularity of the frames involved. This yields the desired conclusion.

To the other identity,

$$\mathfrak{B}\rho_L \circ \mu_{\mathfrak{B}L} = \mathrm{id}_{\mathfrak{B}L},$$

consider the diagram

$$\begin{tikzcd}
 & C\mathfrak{B}L \arrow[dr, "\gamma_{\mathfrak{B}L}"] \arrow[drr, "\beta_{C\mathfrak{B}L}"] & & \\
CL \arrow[ur, "C\beta_L"] \arrow[r, "\gamma_L"] \arrow[dr, "\beta_{CL}"'] & L \arrow[r, "\beta_L"] & \mathfrak{B}L \arrow[r, shift left, "\mu_{\mathfrak{B}L}"] & \mathfrak{B}C\mathfrak{B}LX \arrow[l, shift left, "\mathfrak{B}\rho_L"] \\
 & \mathfrak{B}CL \arrow[ur, "\mathfrak{B}\gamma_L"'] \arrow[urr, "\mathfrak{B}C\beta_L"'] & &
\end{tikzcd}$$

where the squares on the left, the outer squares and the triangles on the right all commute, the latter by the definition of μ and ρ. Now

$$\begin{aligned}
\mathfrak{B}\rho_L \circ \mu_{\mathfrak{B}L} \circ \beta_L \circ \gamma_L &= \mathfrak{B}\rho_L \circ \mu_{\mathfrak{B}L} \circ \gamma_{\mathfrak{B}L} \circ C\beta_L = \mathfrak{B}\rho_L \circ \beta_{C\mathfrak{B}L} \circ C\beta_L = \\
&= \mathfrak{B}\rho_L \circ \mathfrak{B}C\beta_L \circ \beta_{CL} = \mathfrak{B}\gamma_L \circ \beta_{CL} = \beta_L \circ \gamma_L,
\end{aligned}$$

and $\beta_L \circ \gamma_L$ can be cancelled. □

Proposition 3. $\mathfrak{B}$ *induces an equivalence between* $\mathbf{CUniFrm}_{wo}$ *and* $\mathfrak{B}\mathbf{UniFrm}$*, with inverse given by* C*.*

Proof: By general principles, the adjointness produces an equivalence between the full subcategories on either side given by the objects on which the adjunction maps are isomorphisms. Since all μ_M are isomorphisms, this is the entire category on that side. On the other hand, $\rho_L = \gamma_L(C\beta_L)^{-1}$ is an isomorphism iff γ_L is, that is, iff L is complete. □

We now turn to a couple of natural subcategories of $\mathfrak{B}\mathbf{UniFrm}$ and determine what the above equivalence involves for them.

The first of these is the full subcategory given by the totally bounded M, that is, those M for which $\mathfrak{U}M$ is generated by its finite members or, equivalently, whose completion is compact (Banaschewski-Pultr [2]). On the other hand, the compact uniform frames may be identified with the compact regular frames, in view of the fact that each of the latter has a unique uniformity, generated by its finite covers; moreover, any homomorphism between these frames is obviously uniform with respect to these uniformities.

Putting these facts together, we obtain

Proposition 4. $\mathfrak{B}$ *induces an equivalence between the category of compact regular frames and weakly open homomorphisms and the category of totally bounded Boolean uniform frames, with inverse given by* C*.*

In a similar vein, any Boolean frame M may be identified with the uniform frame obtained by equipping it with the uniformity generated by all its finite covers, and the corresponding completion then has the frame $\mathfrak{I}M$ of all ideals of M as its underlying frame. Further, $\mathfrak{I}M$ is a *Gleason* frame, that is, compact, zero-dimensional, and satisfying the Stone identity $x^* \vee x^{**} = e$. On the other hand, for any Gleason frame L, $\mathfrak{B}L$ is a sublattice of L, in virtue of the Stone duality, so that any finite cover of $\mathfrak{B}L$ is actually a cover of L, and hence the unique uniformity of L induces the uniformity determined by all finite covers on $\mathfrak{B}L$.

In all, this shows the following, where $\mathfrak{I}$ is the ideal frame functor.

Proposition 5. $\mathfrak{B}$ *induces an equivalence between the category of Gleason frames and weakly open homomorphisms and the category of all Boolean frames, with inverse given by* $\mathfrak{I}$*.*

Remark 1. Recall from Banaschewski [1] that the *Gleason envelope* of a compact frame L, the counterpart of the familiar Gleason cover of a compact Hausdorff space, is given by the weakly open embedding

$$L \to GL = \mathfrak{I}(\mathfrak{B}L)$$
$$a \mapsto \{x \in \mathfrak{B}L \mid x^* \vee a = e\}.$$

This puts in evidence that the Gleason envelope is functorial for weakly open homomorphisms: $G = \mathfrak{I}\mathfrak{B}$, and that G is the reflection of Gleason frames, a special case of a general result of Johnstone [10]. On the other hand, it provides an interesting representation of the Gleason envelope: in the category of totally bounded Boolean uniform frames, *G corresponds to the functor that enlarges each uniformity to the finite cover uniformity.*

Remark 2. The treatment of the Booleanization of metric frames in Banaschewski-Pultr [3] established the counterpart of Proposition 3 directly, without first proving that completion provides a left adjoint to Booleanization. We note that the proof of Proposition 2 applies to the metric case as well.

Remark 3. One may ask whether there are variants of Proposition 2 in which the weakly open homomorphisms are replaced by a more restricted class of maps. That, however, is not the case: the completion functor produces *all* weakly open uniform homomorphisms between complete uniform frames.

3. Duality

We now want to connect our results on uniform frames with uniform spaces. In particular, we are interested in representing the complete uniform spaces in terms of Boolean uniform frames. By way of preparation, we first establish some general facts concerning the relation between uniform spaces and uniform frames.

To begin with, recall the basic pair of contravariant functors, for the category **UniSp** of uniform spaces and uniformly continuous maps,

$$\mathfrak{O} : \mathbf{UniSp} \to \mathbf{UniFrm}\,, \quad \Sigma : \mathbf{UniFrm} \to \mathbf{UniSp}$$

which come from the corresponding functors for topological spaces and frames: For any uniform space X, the uniform frame $\mathfrak{O}X$ is the frame of open subsets of X, equipped with the uniformity given by the open uniform covers of X. For any uniform frame L, ΣL has the same elements as the frame spectrum of L, that is, the homomorphisms $\xi : L \to \mathbf{2}$, while its uniformity is generated by the covers $\{\Sigma_a \mid a \in A\}$ for $A \in \mathfrak{U}L$, where $\Sigma_a = \{\xi \in \Sigma L \mid \xi(a) = 1\}$. Further, there are the two adjunctions

$$\eta_L : L \to \mathfrak{O}\Sigma L, \quad \eta_L(a) = \Sigma_a$$

and

$$\varepsilon_X : X \to \Sigma\mathfrak{O}X, \quad \varepsilon_X(x) = \widetilde{x}, \quad \widetilde{x}(U) = \mathrm{card}\,(U \cap \{x\}).$$

Note that, for any separated uniform space X, because it is Hausdorff in its uniform topology and therefore sober, ε_X is an isomorphism. On the other hand, η_L is always a surjection of uniform frames, although it may fail very badly to be an isomorphism. For instance, $\Sigma L = \emptyset$ and then $\mathfrak{O}\Sigma L$ is trivial whenever $L = \mathfrak{B}N$ for any uniform frame N which has no atoms. A uniform frame L for which η_L is an isomorphism will be called *spatial*.

Concerning spatial uniform frames, we need the following

Lemma 4. *For any dense surjection $h : L \to K$ with spatial K, $\Sigma h : \Sigma K \to \Sigma L$ is a dense embedding.*

Proof: Since h is onto, Σh is one-one. Further, for any $A \in \mathfrak{U}K$ there exist $B \in \mathfrak{U}L$ such that $h[B] \leq A$, and thus

$$\{(\Sigma h)^{-1}(\Sigma_b) \mid b \in B\} = \{\Sigma_{h(b)} \mid b \in B\} \leq \{\Sigma_a \mid a \in A\},$$

showing that Σh is a uniform subspace embedding. Finally, to see Σh is dense, consider any $\Sigma_a \neq \emptyset$ in ΣL. Then $a \neq 0$, therefore $h(a) \neq 0$ since h is dense, and the fact that K is spatial then ensures that $(\Sigma h)^{-1}(\Sigma_a) = \Sigma_{h(a)}$ is also non-void. This proves the desired result. □

Remark. This lemma no longer holds for non-spatial K: all $\beta_L : L \to \mathfrak{B}L$ are dense surjections but $\Sigma\mathfrak{B}L$ may well be empty.

Next, recall from Banaschewski-Pultr [2] that the spectrum of a complete uniform frame is complete so that one has the contravariant functors

$$C\mathfrak{O} : \mathbf{CUniSp} \to \mathbf{CUniFrm}\,, \quad \Sigma : \mathbf{CUniFrm} \to \mathbf{CUniSp}$$

(prefix $\mathbf{C}$ for completeness), adjoint on the right, with the adjunction maps

$$X \to \Sigma C\mathfrak{O}X = X \xrightarrow{\varepsilon_X} \Sigma\mathfrak{O}X \xrightarrow{\Sigma\gamma_{\mathfrak{O}X}} \Sigma C\mathfrak{O}X$$

and

$$L \to C\mathfrak{O}\Sigma L = L \xrightarrow{\gamma_L^{-1}} CL \xrightarrow{C\eta_L} C\mathfrak{O}\Sigma L,$$

for any complete uniform space X and any complete uniform frame L. In particular, $C\mathfrak{O}$ and Σ induce a dual equivalence between the full subcategories on either side, determined by those objects whose adjunction maps are isomorphisms.

Now, on the space side, every adjunction map is an isomorphism: every ε_X is an isomorphism since complete uniform spaces are understood to be separated, whereas $\Sigma\gamma_{\mathfrak{O}X}$ is always a dense embedding by Lemma 4 and hence also an isomorphism whenever X is complete.

On the other hand, for any complete uniform frame L, the adjunction map is an isomorphism iff $C\eta_L$ is an isomorphism, and by Lemma 2 this holds iff η_L is dense. Now, the latter means that $\Sigma_a \neq \emptyset$ whenever $a \neq 0$; explicitly, this holds iff, for any non-zero $a \in L$, there exists a homomorphism $\xi : L \to \mathbf{2}$ such that $\xi(a) = 1$. Spatial uniform frames obviously satisfy this, but there are quite natural examples of non-spatial uniform frames with this property. We shall call a (uniform) frame *weakly spatial* whenever it satisfies this condition. Hence we now have

Proposition 6. *The category of complete uniform spaces is dually equivalent to the category of weakly spatial complete uniform frames by the contravariant functor $C\mathfrak{O}$, with inverse Σ.*

Calling a uniformly continuous map $f : X \to Y$ weakly open whenever $\mathfrak{O}f : \mathfrak{O}Y \to \mathfrak{O}X$ is weakly open, $C\mathfrak{O}$ takes the corresponding category $\mathbf{CUniSp}_{wo}$ into $\mathbf{CUniFrm}_{wo}$ by Lemma 3. On the other hand, $\Sigma h : \Sigma K \to \Sigma L$ is weakly open for weakly open $h : L \to K$, provided L and K are weakly spatial: If $\Sigma_a \subseteq \Sigma L$ is dense open then $a \in L$ is dense since L is weakly spatial, and hence $(\Sigma h)^{-1}(\Sigma_a) = \Sigma_{h(a)}$ is dense because h is weakly open and K weakly spatial.

As a consequence of this, we also have the following restricted version of the previous proposition:

Proposition 7. *$C\mathfrak{O}$ induces a dual equivalence, with inverse Σ, between $\mathbf{CUniSp}_{wo}$ and the category of weakly spatial complete uniform frames with weakly open uniform homomorphisms.*

Remark. We note that the weak openness of a (uniformly) continuous map has the suggestive topological characterization: $f : X \to Y$ is weakly open if $\operatorname{int} \overline{f[U]} \neq \emptyset$ for any non-void open $U \subseteq X$ (Banaschewski-Pultr [4, Theorem 4.4]).

Combining the results of Propositions 3 and 6 obviously leads to a dual equivalence involving certain Boolean uniform frames. To make this explicit requires an internal characterization of those Boolean uniform frames which have weakly spatial completion. This can be done by means of the *Cauchy spectrum* of uniform frames (Banaschewski-Pultr [2]).

We recall the relevant details. A *Cauchy* filter in a uniform frame L is a filter which meets every uniform cover of L. A *regular* Cauchy filter in L is a Cauchy filter P in L such that, for each $a \in P$, there exist $b \lhd a$ in P, the latter meaning that $Cb \leq a$ for some $C \in \mathfrak{U}L$. Then, the *Cauchy spectrum* ΨL of L is the uniform space whose points are the regular Cauchy filters of L, and whose uniformity is generated by the covers

$$\Psi_A = \{\Psi_a \mid a \in A\}, \quad \Psi_a = \{P \in \Psi L \mid a \in P\} \qquad (A \in \mathfrak{U}L).$$

ΨL is always complete, and the correspondence $L \mapsto \Psi L$ is contravariantly functorial. Moreover, there is a natural equivalence $\lambda : \Sigma C \to \Psi$ such that, for each L, $\lambda_L : \Sigma CL \to \Psi L$ takes any $\xi : CL \to \mathbf{2}$ to the regular Cauchy filter

$$\{a \in L \mid \xi k_L(a) = 1\}$$

where $k_L : L \to CL$ is the right adjoint of $\gamma_L : CL \to L$.

Now we have

Lemma 5. *The following are equivalent for any Boolean uniform frame M:*

(1) *CM is weakly spatial.*
(2) *The Cauchy spectrum of M separates the elements of M.*
(3) *The spectrum of CM separates the elements $k_L(a)$, $a \in M$.*

Proof: (1) $\Rightarrow$ (2). If $a \not\leq b$ in M then $a \wedge b^* \neq 0$ since M is Boolean, hence $k_M(a \wedge b^*) \neq 0$, and therefore there exist $\xi : CM \to \mathbf{2}$ for which $\xi k_M(a \wedge b^*) = 1$. It then follows, for the regular Cauchy filter P corresponding to ξ, that $a \wedge b^* \in P$ and hence $a \in P$ but $b \notin P$.

(2) $\Rightarrow$ (3). If $k_M(a) \not\leq k_M(b)$ for some $a, b \in M$, then also $a \not\leq b$, and if P is a regular Cauchy filter such that $a \in P$ and $b \notin P$ then $\xi(k_M(a)) = 1$ and $\xi(k_M(b)) = 0$ for the $\xi : CM \to \mathbf{2}$ that corresponds to P.

(3) $\Rightarrow$ (1). Given any non-zero c in CM, there exist $a \in CM$ such that $0 < a \lhd c$, by the properties of uniform frames, and hence $a \wedge b = 0$ and $c \vee b = e$ for some $b \in CM$. It then follows that $k_M \gamma_M(a) \wedge b = 0$, since $\gamma_M k_M = \mathrm{id}$ and γ_M is dense, and therefore $k_M \gamma_M(a) \leq c$. Now, there exist $\xi : CM \to \mathbf{2}$ for which $\xi(k_M \gamma_M(a)) = 1$ by hypothesis, and then also $\xi(c) = 1$. □

We shall call uniform frames with the property (2) in the above lemma *Cauchy spatial*.

Putting together Propositions 6 and 3 we now conclude:

Proposition 8. $\mathfrak{BO}$ *induces a dual equivalence, with inverse* Ψ*, between the category of complete uniform spaces with weakly open uniformly continuous maps and the category of Cauchy spatial Boolean uniform frames.*

There is a special case of this result which merits separate mention. Recall that a uniform frame L is said to be *of countable type* if the filter $\mathfrak{U}L$ has a countable basis. Countable type permits an inductive procedure, due to Isbell [8], which shows that any uniform frame of this kind is Cauchy spatial. Hence the following

Corollary. $\mathfrak{BO}$ *induces a dual equivalence, with inverse* Ψ*, between the category of complete uniform spaces of countable type with weakly open uniformly continuous maps and the category of Boolean uniform frames of countable type.*

Remark. Any Boolean uniform frame M with the finite cover uniformity is Cauchy spatial (assuming the Boolean Ultrafilter Theorem) since the regular Cauchy filters of such M are just the ultrafilters. Hence, the duality of the preceding proposition includes the duality between extremally disconnected Boolean spaces and complete Boolean algebras, which, in turn, is contained in classical Stone Duality between all Boolean spaces and Boolean algebras. Thus, Proposition 7 may be viewed as an extension of a part of Stone Duality to arbitrary complete uniform spaces.

Acknowledgements. An earlier version of this paper was in part written while the first author was visiting the university of Cape Town on sabbatical leave from McMaster University. Financial assistance from the Natural Sciences and Engineering Research Council of Canada, the Categorical Topology Research group at the University of Cape Town, and the Grant Agency of the Czech Republic under Grant 201/93/950 is gratefully acknowledged.

References

[1] Banaschewski B., *Compact regular frames and the Sikorski Theorem*, Kyungpook J. Math. **28** (1988), 1–14.

[2] Banaschewski B., Pultr A., *Samuel compactification and completion of uniform frames*, Math. Proc. Cambridge Phil. Soc. **108** (1990), 63–78.

[3] ———, *A Stone duality for metric spaces*, Canad. Math. Soc. Conf. Proceedings **13** (1992), 33–42.

[4] ———, *Variants of openness*, Appl. Categ. Structures **2** (1994), 331–350.

[5] ———, *Booleanization*, preprint.

[6] Glivenko V., *Sur quelque points de la logique de M. Brouwer*, Acad. Royal Belg. Bull. Sci. **15** (1929), 183–188.

[7] Herrlich H., Strecker G.E., *H-closed spaces and reflective subcategories*, Math. Annalen **177** (1968), 302–309.

[8] Isbell J.R., *Atomless parts of spaces*, Math. Scand. **31** (1972), 5–32.

[9] Johnstone P.T., *Stone Spaces*, Cambridge University Press, Cambridge, 1982.

[10] ———, *Factorization theorems for geometric morphisms, II.*, Springer Lecture Notes in Math. **915** (1982), 216–233.

[11] Kříž I., *A direct description of uniform completion in locales and a characterization of LT groups*, Cahiers Top. et Géom. Diff. Cat. **27** (1986), 19–34.

[12] Mioduszewski J., Rudolf L., *H-closed and extremally disconnected Hausdorff spaces*, Dissertationes Math. **66** (1969).

[13] Vickers S., *Topology via Logic*, Cambridge Tracts in Theor. Comp. Sci., Number 5, Cambridge University Press, Cambridge, 1985.

Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario L8S 4K1, Canada

Department of Applied Mathematics, Faculty of Mathematics and Physics, Charles University, Malostranské nám. 25, 118 00 Praha 1, Czech Republic

(*Received* May 15, 1995)