Indiscernibles and dimensional compactness

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Abstract. This is a contribution to the theory of topological vector spaces within the framework of the alternative set theory. Using indiscernibles we will show that every infinite set $uS \subseteq G$ in a biequivalence vector space $\langle W, M, G\rangle$, such that $x - y \notin M$ for distinct $x, y \in u$, contains an infinite independent subset. Consequently, a class $X \subseteq G$ is dimensionally compact iff the π-equivalence $\doteq_M$ is compact on X. This solves a problem from the paper [NPZ 1992] by J. Náter, P. Pulmann and the second author.

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This note is a continuation of the papers [ŠZ 1991] and [NPZ 1992]. Its goal is the solution of Problem 2 from [NPZ 1992].

All unexplained terminology and notation concerning biequivalence vector spaces can be found in these papers. For the fundamentals of the alternative set theory and biequivalences the reader is referred to the book [V 1979] and the article [GZ 1985], respectively. For the reader's convenience we explain some of the more frequently occurring notions and results.

A *biequivalence vector space* (BVS for short) is a triple $\langle W, M, G\rangle$ where W is a set-theoretically definable (briefly Sd) vector space over the Sd-field $\mathbb{Q}$ of rationals and the classes M and G (the monad and the galaxy of 0, respectively) are subject to some natural conditions which can be found in [ŠZ 1991].

A class $X \subseteq W$ in a biequivalence vector space $\langle W, M, G\rangle$ is called *separated* if $x - y \in M \;\Rightarrow\; x = y$ for all $x, y \in X$.

$X \subseteq W$ is called *independent* if

$$X \cap M = \emptyset \qquad \text{and} \qquad (\forall x \in X)\big([\{x\}] \cap \big([X \setminus \{x\}] + M\big) \subseteq M\big),$$

where

$$[X] = \Big\{\sum_{i=1}^{n} \alpha_i x^i;\; n \in N,\; \langle \alpha_1, \dots, \alpha_n\rangle \in \mathbb{Q}^n,\; \langle x^1, \dots, x^n\rangle \in X^n\Big\}$$

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denotes the subspace of W (algebraically) generated by X, i.e. formed by all linear combinations (including the infinite ones) of elements of X.

One can readily see that X is independent iff $X \cap M = \emptyset$ and for any subset $u \subseteq X$ and each $x \in u$ the canonical projection $P^u_x \colon [u] \to [\{x\}]$ given by the linear extension of the map $x \mapsto x$ and $y \mapsto 0$ for $y \in u \setminus \{x\}$ is continuous regarded as a linear map between the biequivalence vector subspaces $\big\langle [u], M \cap [u], G \cap [u] \big\rangle$ and $\big\langle [\{x\}], M \cap [\{x\}], G \cap [\{x\}] \big\rangle$ of $\langle W, M, G \rangle$.

It follows that each independent class in $\langle W, M, G \rangle$ is algebraically independent and the class of all independent subsets of W is a σ-class.

For classes $X \subseteq G$ the independence condition can be stated in a more familiar form: namely, $X \subseteq G$ is independent iff for each $n \in \mathbb{N}$ and all $\langle \alpha_1, \dots, \alpha_n \rangle \in \mathbb{Q}^n$, $\langle x^1, \dots, x^n \rangle \in X^n$, such that $x^i \neq x^j$ for $i \neq j$, we have

$$\sum_{i=1}^{n} \alpha_i x^i \in M \;\Rightarrow\; (\forall\, i \leq n)(\alpha_i \doteq 0).$$

For any BVS $\langle W, M, G \rangle$ we can find a hyperreal number p, $0 < \cdot\, p \leq 1$, and a $(p,1)$-valuation $\|\cdot\|$, inducing its biequivalence structure (see [ŠZ 1991]). Roughly speaking, $\|\cdot\|$ is an “Sd-norm” on W, with the homogeneity condition replaced by p-homogeneity, i.e.

$$\|\alpha x\| = |\alpha|^p \|x\|$$

for any $\alpha \in \mathbb{Q}$, $x \in W$.

In proving our result we will make use of the notion of indiscernibles with respect to some countable language of set-theoretical formulas (cf. [SVe 1981]). To start with let us note that there is a canonical linear order $<$ of the universal class $\mathbb{V}$, given by an $\mathrm{Sd}_{\emptyset}$-bijection $\mathbb{V} \cong \mathbb{N}$. Whenever we speak about indiscernibles, we will always have in mind indiscernibles with respect to this natural order of $\mathbb{V}$. Moreover, for any set u we denote $u^{(i)}$ the i-th element of u with respect to this order; i.e. $u = \big\{u^{(1)}, \dots, u^{(n)}\big\}$, where $n = \#\, u$ (the number of elements of u), and u is listed in its canonical order.

Let $c \in \mathbb{V}$ be any constant. A class X will be called a class of c-indiscernibles if it is a class of indiscernibles for the language $\mathbb{FSL}_{\{c\}}$ of all (finite) set-theoretical formulas containing at most one constant, namely c. In other words, X is a class of c-indiscernibles iff for any formula $\varphi(x_1, \dots, x_k)$ of the language $\mathbb{FSL}_{\{c\}}$ and any two k element subsets $u, v \subseteq X$ we have

$$\varphi\big(u^{(1)}, \dots, u^{(k)}\big) \iff \varphi\big(v^{(1)}, \dots, v^{(k)}\big).$$

There are only countably many formulas in the language $\mathbb{FSL}_{\{c\}}$; hence an iterated application of the Ramsey theorem together with the overspill principle (which is a consequence of the prolongation axiom in AST) yield the following result:

Lemma. *For every infinite set u and any constant c there is an infinite set of c-indiscernibles $w \subseteq u$.*

Now, everything is ready to state and prove the announced result.

Theorem. *Let $\langle W, M, G\rangle$ be a biequivalence vector space and $u \subseteq G$ be an infinite separated set. Then there is an infinite independent subset $v \subseteq u$.*

PROOF: Let c be a constant such that both the vector space $\langle W, +, \cdot, 0\rangle$ (i.e. the class W as well as the addition $W \times W \to W$ and the scalar multiplication $\mathbb{Q} \times W \to W$) and the $(p,1)$-valuation $\|\cdot\|$ inducing the biequivalence structure of $\langle W, M, G\rangle$ are definable by formulas of the language $\mathbb{FSL}_{\{c\}}$. Fix a positive constant $\gamma \in \mathbb{BQ}$ such that $\|x\| \leq \gamma$ for each $x \in u$.

By the Lemma, there is an infinite subset $w \subseteq u$ of c-indiscernibles. We will show that the desired independent set v can be chosen as certain infinite subset of w.

For each $k \in \mathbb{N}^+ = \mathbb{N} \setminus \{0\}$ let

$$J_k = \left\{ \frac{j}{k^r};\ j \in \mathbb{Z},\ |j| \leq k^r \right\},$$

where $r = 1 + \lceil 1/p \rceil$ ($\lceil a \rceil$ is the least integer $\geq a$). Obviously, J is an $\mathrm{Sd}_\emptyset$-function and, as $p \cdot\!> 0$, we have $1 < r \in \mathbb{FN}$. Further, for each $n \in \mathbb{FN}^+ = \mathbb{FN} \setminus \{0\}$ we put

$$R_n = \Big\{ k \in \mathbb{N}^+;\ (\forall\, \alpha \in {J_k}^k)(\forall\, s, t \in \mathcal{P}_k(w)) \\ \Big(\Big\| \sum_{i=1}^{k} \alpha_i s^{(i)} \Big\| < \frac{1}{n} \iff \Big\| \sum_{i=1}^{k} \alpha_i t^{(i)} \Big\| < \frac{1}{n} \Big) \Big\},$$

where $\mathcal{P}_k(w)$ denotes the set of all k element subsets of w. Then each R_n is an $\mathrm{Sd}_{\{c\}}$-class of natural numbers. As w is a set of c-indiscernibles and for any fixed $k, n \in \mathbb{FN}^+$, $\alpha \in {J_k}^k$ the k-ary relation

$$\varphi(x^1, \dots, x^k) \iff \Big\| \sum_{i=1}^{k} \alpha_i x^i \Big\| < \frac{1}{n}$$

can be expressed by a formula of the language $\mathbb{FSL}_{\{c\}}$, we have $\mathbb{FN}^+ \subseteq R_n$ for each $n \in \mathbb{FN}^+$. By overspill there is an infinite $k \in \mathbb{N}$ such that even $\{1, 2, \dots, k\} \subseteq R_n$ holds for each $n \in \mathbb{FN}^+$. Thus we can assume $2k \leq \#\, w$. Let v be the k element subset $\{w^{(1)}, w^{(3)}, \dots, w^{(2k-1)}\}$ of w. In other words, $v^{(i)} = w^{(2i-1)}$ for $1 \leq i \leq k$. Then $v \subseteq u$ is an infinite set of c-indiscernibles and we claim that it is already independent.

To prove the last claim, take an arbitrary $\alpha \in \mathbb{Q}^k$ such that

$$\sum_{i=1}^{k} \alpha_i v^{(i)} \in M.$$

It suffices to show $\alpha_i \doteq 0$ for each i. Suppose otherwise. Let $j \le k$ be an index such that $|\alpha_j| = \max_{1 \le i \le k} |\alpha_i|$. Then α_j is not infinitesimal, so that $1/\alpha_j \in \mathbb{BQ}$ and

$$\frac{1}{\alpha_j} \sum_{i=1}^{k} \alpha_i v^{(i)} = v^{(j)} + \sum_{i \ne j} \frac{\alpha_i}{\alpha_j} v^{(i)} \in M.$$

As $|\alpha_i/\alpha_j| \le 1$, there is a $\beta \in {J_k}^k$ such that

$$\left| \frac{\alpha_i}{\alpha_j} - \beta_i \right| < \frac{1}{k^r}$$

for $i \ne j$. Then we have

$$\begin{aligned} \Big\| \sum_{i \ne j} \Big(\frac{\alpha_i}{\alpha_j} - \beta_i \Big) v^{(i)} \Big\| &\le \sum_{i \ne j} \Big\| \Big(\frac{\alpha_i}{\alpha_j} - \beta_i \Big) v^{(i)} \Big\| = \sum_{i \ne j} \Big| \frac{\alpha_i}{\alpha_j} - \beta_i \Big|^p \big\| v^{(i)} \big\| \\ &< \frac{(k-1)\gamma}{k^{rp}} < \frac{k\gamma}{k^{(1+1/p)p}} = \frac{\gamma}{k^p} \doteq 0, \end{aligned}$$

as $\gamma \in \mathbb{BQ}$, $p \mathrel{\cdot\!>} 0$ and $k \in \mathbb{N} \setminus \mathbb{FN}$. Let $x = v^{(j)} = w^{(2j-1)}$ and $y = w^{(2j)}$. We have

$$\Big\| x + \sum_{i \ne j} \beta_i v^{(i)} \Big\| \le \Big\| x + \sum \frac{\alpha_i}{\alpha_j} v^{(j)} \Big\| + \Big\| \sum \Big(\frac{\alpha_i}{\alpha_j} - \beta_i \Big) v^{(i)} \Big\| \doteq 0,$$

i.e.,

$$\Big\| x + \sum_{i \ne j} \beta_i v^{(i)} \Big\| < \frac{1}{n}$$

for each $n \in \mathbb{FN}^+$.

However, as w is a set of c-indiscernibles and the positions of x in v and of y in $(v \setminus \{x\}) \cup \{y\}$ with respect to the canonical order are the same, by the choice of k we have

$$\Big\| y + \sum_{i \ne j} \beta_i v^{(i)} \Big\| < \frac{1}{n}$$

for each $n \in \mathbb{FN}^+$, as well. This is the same as

$$y + \sum_{i \ne j} \beta_i v^{(i)} \in M.$$

Hence, $x - y \in M$, contradicting the separateness of u. $\square$

Remark. Note that since the system of all independent subsets of W is a σ-class which is not an Sd-class, one cannot expect to find an "elementary" proof of the Theorem. Using the fact that u is uniformly bounded and separated one can try to construct by induction a sequence $\langle v_n;\ n \in \mathbb{FN}\rangle$ of independent subsets of u such that $v_n \subseteq v_{n+1}$ and $\# v_n = n$ for each n. Then he could look for the infinite independent set $v \subseteq u$ within some set prolongation $\langle v_0, v_1, \dots, v_m\rangle$ of the original sequence, where $m \in \mathbb{N} \setminus \mathbb{FN}$, $v_l \subseteq u$ and $\# v_l = l$ for each $l \leq m$. However, this straightforward method need not yield an independent set v_l for any infinite $l \leq m$, as the class of all independent subsets of u is not revealed.

Recall from [NPZ 1992] that a class X in a BVS $\langle W, M, G\rangle$ is called *dimensionally compact* if every independent subset $u \subseteq X$ is finite. Similarly, we say that the π-equivalence $\doteq_M$ on W, given by

$$x \doteq_M y \iff x - y \in M,$$

is *compact* on a class $X \subseteq W$ if every separated set $u \subseteq X$ is finite (cf. [V 1979], [GZ 1985]).

Corollary. *Let $\langle W, M, G\rangle$ be any biequivalence vector space. A class $X \subseteq G$ is dimensionally compact if and only if the π-equivalence $\doteq_M$ is compact on X.*

There is also a standard version of the Theorem which can be formulated as a certain kind of Ramsey type selection principle within the framework of quasi-normed spaces. This result leads to several issues reaching beyond the natural scope of the present article, so it will be published in a separate paper.

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