Pointwise estimates of nonnegative subsolutions of quasilinear elliptic equations at irregular boundary points

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Abstract. Let u be a weak solution of a quasilinear elliptic equation of the growth p with a measure right hand term μ. We estimate $u(z)$ at an interior point z of the domain Ω, or an irregular boundary point $z \in \partial\Omega$, in terms of a norm of u, a nonlinear potential of μ and the Wiener integral of $\mathbf{R}^n \setminus \Omega$. This quantifies the result on necessity of the Wiener criterion.

Keywords: elliptic equations, Wiener criterion, nonlinear potentials, measure data

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1. Introduction

We study quasilinear elliptic equations of type

$$-\operatorname{div}\mathbf{A}(x,u,\nabla u)+\mathbf{B}(x,u,\nabla u)=\mu\,, \tag{1.1}$$

where $\mathbf{A}$ and $\mathbf{B}$ are Carathéodory functions (precise conditions depending on a growth exponent $p \in (1,\infty)$ will be given later) and $\mu \in (W_0^{1,p}(\Omega))^*$ is a nonnegative Radon measure. We refer to (1.1_0) if $\mu = 0$.

The model equation for (1.1) is

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u)+\lambda|u|^{p-2}u=\mu, \tag{1.2}$$

with $\lambda \in \mathbf{R}$. Sometimes we mention monotone type equations, by this we will understand equations satisfying the structure conditions of [13] (unweighted case). These equations satisfy additional assumptions which guarantee existence and uniqueness results.

We will work with the integrals

$$\mathbf{w}_p(x,E)=\int_0^{r_0}\Big(\frac{\operatorname{cap}_p(E\cap B(x,r),r)}{r^{n-p}}\Big)^{1/(p-1)}\,\frac{dr}{r} \tag{1.3}$$

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and

$$\mathbf{W}_p^\mu(x) = \int_0^{r_0} \Big(\frac{\mu(B(x,r))}{r^{n-p}}\Big)^{1/(p-1)} \frac{dr}{r}. \tag{1.4}$$

The function $\mathbf{W}_p^\mu$ is a kind of nonlinear potential of the measure μ. These potentials were introduced by Adams and Meyers [3], Hedberg [9] and Hedberg and Wolff [10]. For more information on $\mathbf{W}_p$ potentials, we refer to the recent monograph by Adams and Hedberg [2].

We present pointwise estimates for subsolutions of (1.1) in terms of $\mathbf{W}_p^\mu$ and $\mathbf{w}_p(\cdot, \mathbf{R}^n \setminus \Omega)$.

In the interior case, and with $\mu = 0$, the presented estimate is a version of the Trudinger's Harnack inequality for subsolutions [27]. The interior estimate with a nontrivial μ has been proved for monotone type equations by Kilpeläinen and Malý [16]. Notice that lower interior estimates for supersolutions of (1.1) in terms of $\mathbf{W}_p^\mu$, generalizing Trudinger's Harnack inequality for supersolutions, are also valid, see Kilpeläinen and Malý [14] (for monotone type equations), Malý [20], and Malý and Ziemer [23]. Related, but different results are due to Rakotoson and Ziemer [25], Lieberman [17] and Adams [1].

Let $u_0 \in W^{1,p}(\Omega)$ and u be a solution of (1.1_0). We say that u solves the Dirichlet problem with the boundary data u_0 if $u - u_0 \in W_0^{1,p}(\Omega)$. A point $z \in \partial\Omega$ is said to be regular for the equation (1.1_0) if

$$\lim_{x\to z,\, x\in\Omega} u(x) = u_0(z)$$

whenever $u \in \mathcal{C}(\Omega)$ is a solution of the Dirichlet problem with boundary data $u_0 \in W^{1,p}(\Omega) \cap \mathcal{C}(\overline{\Omega})$. Wiener [28] showed that z is regular for the Laplace equation if and only if the classical Wiener criterion is satisfied. This more or less says that z is regular for the Laplace equation if and only if the Wiener integral $\mathbf{w}_2(z, \mathbf{R}^n \setminus \Omega)$ diverges. Littman, Stampacchia, Weinberger [19] proved that the same condition applies to linear elliptic divergence form equations with discontinuous bounded measurable coefficients. If $p \neq 2$, we say that the Wiener condition is satisfied at z if $\mathbf{w}_p(z, \mathbf{R}^n \setminus \Omega)$ diverges, i.e. if $\mathbf{R}^n \setminus \Omega$ is not p-thin at Ω. Maz'ya [21] established the sufficiency of the Wiener criterion under simpler structure assumptions. Gariepy and Ziemer [8] proved the sufficiency in the general case of equation (1.1_0).

The Wiener criteria established by Wiener [28] and Littman, Stampacchia, Weinberger [19] were presented as both necessary and sufficient. On the other hand, the sufficient condition by Maz'ya [21] waited a longer time for its necessity counterpart. For a special class of equations, some necessary conditions differing in an exponent from the sufficient conditions were proved by Skrypnik [26]. The necessity of the Wiener condition for equations of the monotone type was shown by Lindqvist and Martio [18] and Heinonen and Kilpeläinen [11] with the restriction $p > n - 1$. For all $p \in (1, \infty)$, it was proved by Kilpeläinen and Malý in [16].

The estimate given in the present paper implies in some sense the necessity of the Wiener criterion for equations of type (1.1_0) and quantifies the pointwise behavior of solutions at irregular points.

For a wider information about the topic we refer to the prepared monograph [23] by Malý and Ziemer. For consequences and relations to A-superharmonic functions in nonlinear potential theory we refer also to the papers by Kilpeläinen and Malý [16], Heinonen, Kilpeläinen and Martio [12] and to the monograph [13] by Heinonen, Kilpeläinen and Martio.

2. Preliminaries

In what follows, Ω is an open subset of $\mathbf{R}^n$ and p is an exponent in $(1, n]$. We write C, C' etc. for various constants (they may differ from line to line). We denote by $B(z,r)$ the open ball in $\mathbf{R}^n$ with center at z and radius r. If $B = B(z,r)$, then $2B$ means the ball $B(z, 2r)$. We denote by $\mathcal{C}_c^\infty(\Omega)$ the set of all infinitely differentiable functions with a compact support in Ω. The norm in the Lebesgue space $L^p(\Omega)$, resp. in the Sobolev space $W^{1,p}(\Omega)$ is denoted by $\|...\|_p$, resp. $\|...\|_{1,p}$. We use $|E|$ for the Lebesgue measure of the set E.

We define the *p-capacity* of a set $E \subset \mathbf{R}^n$ by $\operatorname{cap}_p E = \operatorname{cap}_p(E, 1)$, where

$$\begin{aligned}\operatorname{cap}_p(E,r) = \inf\Big\{\int_{\mathbf{R}^n} \big(|\nabla\varphi|^p + r^{-p}|\varphi|^p\big)\,dx \colon \varphi \in W^{1,p}(\mathbf{R}^n),\\ \varphi \geq 1 \text{ on an open set containing } E\Big\}\end{aligned}$$

This scale of capacities is natural in connection with the Wiener criterion; for $E \subset B$ it is equivalent to the "condenser capacity" of E w.r.t. $2B$, cf. [13].

A set $U \subset \mathbf{R}^n$ is said to be *p-quasiopen* if for each $\varepsilon > 0$ there is an open set $G \subset \mathbf{R}^n$ such that $\operatorname{cap}_p G < \varepsilon$ and $U \cup G$ is open. Similarly, a function u is said to be *p-quasicontinuous* on Ω if for each $\varepsilon > 0$ there is an open set $G \subset \mathbf{R}^n$ such that $\operatorname{cap}_p G < \varepsilon$ and $u|\Omega \setminus G$ is continuous.

We use the abbreviation p-q.e. (*p-quasi everywhere*) for the phrase "except a set of p-capacity zero". We say that a set $E \subset \mathbf{R}^n$ is *p-thin* at a point $z \in \mathbf{R}^n$ if the *Wiener integral* $\mathbf{w}_p(z,E)$ converges. The *p-fine closure* adds to every set E the set of all points where E is not p-thin. This introduces the *p-fine topology.*

Notice that every $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ has a p-quasicontinuous representative (see Federer and Ziemer [5], Maz'ya and Khavin [22], Meyers [24], Frehse [6] and that a function u on Ω is p-quasicontinuous if and only if it is p-finely continuous p-q.e. (Fuglede [7], Brelot [4], Hedberg and Wolff [10]).

Due to Poincaré's inequality and approximation arguments,

$$\operatorname{cap}_p(E,r) \leq C \int_{B(x_0,2r)} |\nabla\psi|^p\,dx$$

holds whenever $E \subset B(x_0, r)$, $\psi \in W_0^{1,p}(B(x_0, 2r))$, ψ is p-quasicontinuous and $\psi \geq 1$ p-q.e. on E.

Now, let us state our assumptions concerning the equation (1.1). We suppose that the functions $\mathbf{A}\colon \mathbf{R}^n\times\mathbf{R}\times\mathbf{R}^n \to \mathbf{R}^n$ and $\mathbf{B}\colon \mathbf{R}^n\times\mathbf{R}\times\mathbf{R}^n \to \mathbf{R}$ are Borel measurable and satisfy the following structure conditions:

$$
\begin{aligned}
|\mathbf{A}(x,\zeta,\xi)| &\le a_1|\xi|^{p-1} + a_2|\zeta|^{p-1} + a_3,\\
|\mathbf{B}(x,\zeta,\xi)| &\le b_1|\xi|^{p-1} + b_2|\zeta|^{p-1} + b_3 + b_0|\xi|^p,\\
\mathbf{A}(x,\zeta,\xi)\cdot\xi &\ge c_1|\xi|^p - c_2|\zeta|^p - c_3, \quad c_1>0,
\end{aligned}
\tag{2.1}
$$

where a_i, b_i, c_i are nonnegative constants. We write $b = b_0/c_1$. The model example $\mathbf{A}(x,\zeta,\xi) = |\xi|^{p-2}\xi$, $\mathbf{B}(x,\zeta,\xi) = \lambda|\zeta|^{p-2}\zeta$ leads to (1.2).

We say that u is a *subsolution* (frequently termed a "weak subsolution") of (1.1) in Ω if $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, u is p-quasicontinuous (i.e. we admit p-quasicontinuous representatives only) and

$$
\int_\Omega \Big(\mathbf{A}(x,u,\nabla u)\cdot\nabla\varphi + \mathbf{B}(x,u,\nabla u)\,\varphi\Big)\,dx \le \int_\Omega \varphi\,d\mu
\tag{2.2}
$$

holds for all nonnegative "test functions" $\varphi \in \mathcal{C}^\infty_c(\Omega)$. Similarly we define *solutions* using the equality sign.

3. Main estimate

We consider an exponent

$$\gamma \in (p-1, n(p-1)/(n-p+1))$$

and write

$$\tau = \frac{\gamma}{p-1}, \qquad q = \frac{p\gamma}{p-\tau}.$$

Notice that $\tau > 1$ and $q > p$. Let Ω be an open set and $R_0 > 0$ a fixed radius. We consider a fixed equation of type (1.1). We will denote by C a general constant (not necessarily the same at different occurrences) depending only on n, p, γ, R_0, on the upper bound of $b_0 u$ and on the structure constants.

3.1 Lemma. *Let $u \in W^{1,p}(\Omega)$ be a subsolution of $-\operatorname{div}\mathbf{A}+\mathbf{B} = \mu$ in Ω. Suppose that either u is upper bounded or $b_0 = 0$. Let $\ell \in [0,\infty)$, Φ be a nonnegative bounded Borel measurable function on $\mathbf{R}$ which vanishes on $(-\infty,\ell)$ and λ be the L^1-norm of Φ. Let $\omega \in W^{1,p}_0(\Omega)$, $0 \le \omega \le 1$. Then*

$$
\begin{aligned}
&\int_\Omega \Phi(u)\,|\nabla u|^p\,\omega^p\,dx\\
&\quad\le C\int_{\Omega\cap\{u>\ell\}} \Phi(u)(1+u^p)\,\omega^p\,dx\\
&\qquad + C\lambda\left(\int_{\Omega\cap\{u>\ell\}} \big(|\nabla u|^{p-1} + u^{p-1} + 1\big)\,\omega^{p-1}(\omega + |\nabla\omega|)\,dx + \mu(\{\omega>0\})\right).
\end{aligned}
$$

Proof: We write

$$\Psi(t) = \int_0^t \Phi(s)\, ds,$$
$$L = \Omega \cap \{u > \ell\}.$$

Using the test function

$$\varphi = \Psi(u)\, e^{bu}\, \omega^p$$

with

$$\begin{aligned}\nabla\varphi = {} & \Phi(u)\, \nabla u\, e^{bu}\, \omega^p \\ & + b\, \Psi(u)\, \nabla u\, e^{bu}\, \omega^p \\ & + p\, \Psi(u)\, e^{bu}\, \omega^{p-1} \nabla\omega\end{aligned}$$

we obtain

$$\begin{aligned}& \int_L \mathbf{A}(x,u,\nabla u)\cdot\nabla u\, \Phi(u)\; e^{bu}\, \omega^p\, dx \\ & + b \int_L \mathbf{A}(x,u,\nabla u)\cdot\nabla u\, \Psi(u)\; e^{bu}\, \omega^p\, dx \\ & + p \int_L \mathbf{A}(x,u,\nabla u)\cdot \Psi(u)\, e^{bu}\, \omega^{p-1}\nabla\omega\, dx \\ & + \int_L \mathbf{B}(x,u,\nabla u)\, \Psi(u)\, e^{bu}\, \omega^p\, dx \\ & \quad \le \int_L \Psi(u)\, e^{bu}\, \omega^p\, d\mu.\end{aligned} \tag{3.1}$$

Taking the structure into account, we get

$$\begin{aligned}& \int_L \mathbf{A}(x,u,\nabla u)\cdot\nabla u\, \Phi(u)\, e^{bu}\, \omega^p\, dx \\ & \quad \ge \int_L \big(c_1\, |\nabla u|^{\,p} - c_2 u^p - c_3\big)\, \Phi(u)\, e^{bu}\, \omega^p\, dx\,,\end{aligned} \tag{3.2}$$

$$\begin{aligned}& - b \int_L \mathbf{A}(x,u,\nabla u)\cdot\nabla u\, \Psi(u)\, e^{bu}\, \omega^p\, dx \\ & \quad \le -bc_1 \int_L |\nabla u|^{\,p}\, \Psi(u)\, e^{bu}\, \psi^p \eta^p\, dx \\ & \qquad + b \int_L \big(c_2 u^p + c_3\big)\Psi(u)\, e^{bu}\, \omega^p\, dx\,,\end{aligned} \tag{3.3}$$

$$\begin{aligned}& - \int_L \mathbf{A}(x,u,\nabla u)\cdot \Psi(u)\, e^{bu}\, \omega^{p-1}\nabla\omega\, dx \\ & \le \int_L \Big(a_1\, |\nabla u|^{\,p-1} + a_2 u^{p-1} + a_3\Big)\, \Psi(u)\, e^{bu}\, \omega^{p-1}\nabla\omega\, dx\,,\end{aligned} \tag{3.4}$$

and

$$
\begin{aligned}
& -\int_L \mathbf{B}(x,u,\nabla u)\,\Psi(u)\,e^{bu}\,\omega^p\,dx \\
& \le \int_L \Big(b_1\,|\nabla u|^{\,p-1} + b_2 u^{p-1} + b_3\Big)\,\Psi(u)\,e^{bu}\,\omega^p\,dx \\
& \quad + b_0 \int_L |\nabla u|^{\,p}\,\Psi(u)\,e^{bu}\,\omega^p\,dx.
\end{aligned} \tag{3.5}
$$

From (3.1)–(3.5) we obtain

$$
\begin{aligned}
& c_1 \int_L \Phi(u)\,|\nabla u|^{\,p}\,e^{bu}\,\omega^p\,dx \\
& \quad + bc_1 \int_L \Psi(u)\,|\nabla u|^{\,p}\,e^{bu}\,\omega^p\,dx \\
& \le \int_L \Phi(u)\Big(c_2 u^p + c_3\Big)\,e^{bu}\,\omega^p\,dx \\
& \quad + \int_L \Psi(u)\Big(p\,\big(a_1\,|\nabla u|^{\,p-1} + a_2 u^{p-1} + a_3\big)\,|\nabla\omega| \\
& \quad + \big(b_1\,|\nabla u|^{\,p-1} + (c_2 bu + b_2)u^{p-1} + c_3 b + b_3\big)\,\omega\Big)\,e^{bu}\,\omega^{p-1}\,dx \\
& \quad + b_0 \int_L \Psi(u)\,|\nabla u|^{\,p}\,e^{bu}\,\omega^p\,dx \\
& \le \int_L \Psi(u)\omega^p\,d\mu.
\end{aligned} \tag{3.6}
$$

Since $b_0 = bc_1$, $bu \le C$, $\omega \le 1$ and $\Psi \le \lambda$, it follows

$$
\begin{aligned}
& \int_L \Phi(u)\,|\nabla u|^{\,p}\,\omega^p\,dx \\
& \quad \le C \int_L \Phi(u)(1+u^p)\,\omega^p\,dx \\
& \qquad + C\lambda\,\Big(\int_{\Omega\cap\{u>\ell\}} \big(\,|\nabla u|^{\,p-1} + u^{p-1} + 1\big)\;\omega^{p-1}(\omega + |\nabla\omega|)\,dx + \mu(\{\omega>0\})\Big)
\end{aligned}
$$

as required. $\square$

3.2 Lemma. *Let $u \in W^{1,p}(\Omega)$ be a subsolution of $-\operatorname{div}\mathbf{A}+\mathbf{B} = \mu$ in Ω. Suppose that either u is upper bounded or $b_0 = 0$. Let $B = B(x_0,r)$, where $0 < r < R_0$, be an open ball in $\mathbf{R}^n$. Let $\eta, \varphi, \psi \in W^{1,p}(B)$. Suppose that $0 \le \eta \le 1$, $0 \le \varphi \le 1$, $0 \le \psi \le 1$, $\eta\psi \in W^{1,p}(B\cap\Omega)$, $(1-\varphi)(1-\psi) = 0$ and $\nabla\eta \le 5/r$. Suppose that $\ell \ge 0$.*

(a) *If* $\delta > 0$, *then*

$$\int_L |\nabla w_\delta|^p\,dx \le C\Big(\delta^{-p} r^n (1+\ell^p) + r^{-p}\int_{B\cap\{u>\ell\}\cap\{\varphi<1\}} \Big(1+\frac{u-\ell}{\delta}\Big)^\gamma dx + \delta^{1-p}\,\mu(B(x_0,r)) + \delta^{1-p}(1+\|u\|_\infty)^{p-1}\int_B (r^{-p}\varphi^p + |\nabla\varphi|^p + |\nabla\psi|^p)\,dx\Big),$$

where

$$w_\delta = \Big(\Big(1+\frac{(u-\ell)^+}{\delta}\Big)^{\gamma/q} - 1\Big)\psi\eta.$$

(b) *There is a constant* $\kappa > 0$, *depending only on* n, p, γ, R_0, *on the upper bound of* $b_0 u$ *and on the structure constants, such that*

$$\Big(r^{-n}\int_{B\cap\Omega\cap\{u>\ell\}} (u-\ell)^\gamma \psi^q \eta^q\,dx\Big)^{(p-1)/\gamma} \le C\Big(r^{p-1}(1+\ell)^{p-1} + r^{p-n}\,\mu(B(x_0,r)) + (1+\|u\|_\infty)^{p-1}\,r^{p-n}\int_B (r^{-p}\varphi^p + |\nabla\varphi|^p + |\nabla\psi|^p)\,dx\Big),$$

provided that

$$|B\cap\{u>\ell\}\cap\{\varphi<1\}| \le (2r)^n\kappa \tag{3.7}$$

and

$$\int_{B\cap\{u>\ell\}\cap\{\varphi<1\}} (u-\ell)^\gamma\,dx \le 2^{n+\gamma}\int_{B\cap\Omega\cap\{u>\ell\}} (u-\ell)^\gamma\psi^q\eta^q\,dx. \tag{3.8}$$

Proof: (a) We write

$$\begin{aligned}
\omega &= \psi\eta,\\
\sigma &= \omega\varphi,\\
v &= \frac{(u-\ell)^+}{\delta},\\
M &= 1+\|u\|_\infty,\\
L &= B\cap\Omega\cap\{u>\ell\},\\
E &= L\cap\{\varphi<1\},\\
F &= L\cap\{\varphi=1\}.
\end{aligned}$$

Note that

$$w_\delta = \big((1+v)^{\gamma/q} - 1\big)\omega,$$

$$\nabla w_\delta = \frac{\gamma}{q}(1+v)^{-\tau/p}\nabla v\ \omega + \big((1+v)^{\gamma/q} - 1\big)\,\nabla\omega.$$

Since

$$\begin{aligned}
&\big((1+v)^{\gamma/q} - 1\big)^p \le C\min(v^{p-\tau}, v^p) \le C\min\big((1+v)^\gamma, v^{p-1}\big),\\
&v^{p-1} \le \delta^{1-p}u^{p-1} \le \delta^{1-p}M^{p-1},\\
&\omega = \eta \text{ on } E,\\
&\omega = \sigma \text{ on } F,
\end{aligned} \tag{3.9}$$

it follows

$$\begin{aligned}
&\int_L |\nabla w_\delta|^p\,dx\\
&\quad\le C\Big(\int_E (1+v)^\gamma|\nabla\eta|^p\,dx + M^{p-1}\delta^{1-p}\int_F |\nabla\sigma|^p\,dx\Big)\\
&\qquad + \delta^{-p}\int_L (1+v)^{-\tau}\,|\nabla u|^{\,p}\,\omega^p\,dx.
\end{aligned} \tag{3.10}$$

We use Lemma 3.1 with

$$\Phi(t) = \begin{cases} (1+\frac{(t-\ell)^+}{\delta})^{-\tau}, & t > \ell,\\ 0, & t \le \ell.\end{cases}$$

Then the L^1-norm of Φ is bounded by $(\tau-1)^{-1}\,\delta$. We get

$$\begin{aligned}
&\int_L (1+v)^{-\tau}\,|\nabla u|^{\,p}\,\omega^p\,dx\\
&\quad\le C\int_L (1+v)^{-\tau}(1+u^p)\,\omega^p\,dx\\
&\qquad + C\delta\Big(\int_L \big(\,|\nabla u|^{\,p-1} + u^{p-1} + 1\big)\ \omega^{p-1}(\omega + |\nabla\omega|)\,dx + \mu(B)\Big).
\end{aligned} \tag{3.11}$$

We estimate

$$\begin{aligned}
(1+u^p)(1+v)^{-\tau} &\le (1+u^p)(1+v)^{-1} \le C(1+\ell^p+\delta^p v^p)(1+v)^{-1}\\
&\le C(1+\ell^p+\delta^p v^{p-1})
\end{aligned}$$

Using (3.9) it follows

$$\begin{aligned}
&\int_L (1+v)^{-\tau}(1+u^p)\,\omega^p\,dx\\
&\quad\le Cr^n(1+\ell^p) + \delta^p\int_L v^{p-1}\omega^p\,dx\\
&\quad\le C\Big(r^n(1+\ell^p) + \delta M^{p-1}\int_F \sigma^p\,dx + \delta^p\int_E (1+v)^\gamma\omega^p\,dx\Big).
\end{aligned} \tag{3.12}$$

Choose $\varepsilon > 0$. Young's inequality yields

$$(3.13)\qquad \begin{aligned}&(1+u^{p-1}+|\nabla u|^{p-1})\omega^{p-1}(\omega+|\nabla\omega|)\\ &\quad\le C\frac{\varepsilon}{\delta}(1+v)^{-\tau}(1+u^p+|\nabla u|^p)\omega^p + C\Big(\frac{\varepsilon}{\delta}\Big)^{1-p}(1+v)^{\gamma}(\omega^p+|\nabla\omega|^p).\end{aligned}$$

Recall that $\omega = \eta$ on E. We infer from (3.13) that

$$(3.14)\qquad \begin{aligned}&\int_E\big(|\nabla u|^{p-1}+u^{p-1}+1\big)\ \omega^{p-1}(\omega+|\nabla\omega|)\,dx\\ &\quad\le C\frac{\varepsilon}{\delta}\int_L(1+v)^{-\tau}(1+u^p+|\nabla u|^p)\omega^p\,dx\\ &\qquad + C\Big(\frac{\varepsilon}{\delta}\Big)^{1-p}\int_E(1+v)^{\gamma}(\eta^p+|\nabla\eta|^p)\,dx.\end{aligned}$$

Now, we will estimate the integration on F. We use Lemma 3.1 again with Φ being the characteristic function of the interval $[\ell, M]$ and with σ instead of ω. Then the L^1-norm of Φ is bounded by M and we get

$$(3.15)\qquad \begin{aligned}&\int_L|\nabla u|^p\,\sigma^p\,dx\\ &\quad\le CM\Big(\int_L\big(|\nabla u|^{p-1}+u^{p-1}+1\big)\ \sigma^{p-1}(\sigma+|\nabla\sigma|)\,dx+\mu(B)\Big)\\ &\qquad + C\int_L(1+u^p)\,\sigma^p\,dx\\ &\quad\le CM^p\int_B(\sigma^p+|\nabla\sigma|^p)\,dx\\ &\qquad + CM\Big(\int_L|\nabla u|^{p-1}\sigma^{p-1}(\sigma+|\nabla\sigma|)\,dx+\mu(B)\Big).\end{aligned}$$

Choose $\varepsilon_1 > 0$. A use of Young's inequality yields

$$(3.16)\qquad \begin{aligned}&|\nabla u|^{p-1}\,\sigma^{p-1}(\sigma+|\nabla\sigma|)\\ &\quad\le\frac{\varepsilon_1}{M}\ |\nabla u|^p\sigma^p + C\Big(\frac{\varepsilon_1}{M}\Big)^{1-p}(\sigma^p+|\nabla\sigma|^p).\end{aligned}$$

From (3.15) and (3.16) we get

$$(3.17)\qquad \begin{aligned}&\int_L\big(|\nabla u|^{p-1}+u^{p-1}+1\big)\ \sigma^{p-1}(\sigma+|\nabla\sigma|)\,dx\\ &\quad\le CM^{p-1}\int_B(\sigma^p+|\nabla\sigma|^p)+\int_L|\nabla u|^{p-1}\sigma^{p-1}(\sigma+|\nabla\sigma|)\,dx\\ &\quad\le C(1+\varepsilon_1^{1-p})M^{p-1}\int_B(\sigma^p+|\nabla\sigma|^p)\,dx + C\frac{\varepsilon_1}{M}\int_L|\nabla u|^p\sigma^p\,dx\\ &\quad\le C(1+\varepsilon_1+\varepsilon_1^{1-p})M^{p-1}\int_B(\sigma^p+|\nabla\sigma|^p)\,dx\\ &\qquad + C\varepsilon_1\int_L|\nabla u|^{p-1}\sigma^{p-1}(\sigma+|\nabla\sigma|)\,dx + C\varepsilon_1\mu(B).\end{aligned}$$

Using ε_1 small enough, by a cancellation we obtain

$$\begin{aligned}&\int_L \big(|\nabla u|^{\,p-1} + u^{p-1} + 1\big)\ \sigma^{p-1}(\sigma + |\nabla\sigma|)\, dx\\ &\qquad \le C\Big(M^{p-1}\int_B (\sigma^p +\ |\nabla\sigma|^{\,p})\, dx + \mu(B)\Big).\end{aligned}$$

As $\sigma = \omega$ on F, it follows

$$\begin{aligned}&\int_F \big(|\nabla u|^{\,p-1} + u^{p-1} + 1\big)\ \omega^{p-1}(\omega + |\nabla\omega|)\, dx\\ &\qquad \le C\Big(M^{p-1}\int_B (\sigma^p +\ |\nabla\sigma|^{\,p})\, dx + \mu(B)\Big).\end{aligned}\tag{3.18}$$

From (3.11), (3.12), (3.13), (3.14) and (3.18) we deduce that

$$\begin{aligned}&\int_L (1+v)^{-\tau}\,|\nabla u|^{\,p}\,\omega^p\, dx\\
&\quad\le C\int_L (1+v)^{-\tau}(1+u^p)\,\omega^p\, dx\\
&\qquad + C\delta\Big(\int_L \big(|\nabla u|^{\,p-1} + u^{p-1} + 1\big)\ \omega^{p-1}(\omega + |\nabla\omega|)\, dx + \mu(B)\Big)\\
&\quad\le C\varepsilon\int_L (1+v)^{-\tau}\,|\nabla u|^{\,p}\,\omega^p\, dx\\
&\qquad + C(1+\varepsilon)\int_L (1+v)^{-\tau}(1+u^p)\,\omega^p\, dx\\
&\qquad + C\delta^p\varepsilon^{1-p}\int_E (1+v)^{\gamma}(\eta^p + |\nabla\eta|^p)\, dx\\
&\qquad + C\delta\mu(B) + \delta M^{p-1}\int_L (\sigma^p + |\nabla\sigma|^p)\, dx\\
&\quad\le C\varepsilon\int_L (1+v)^{-\tau}\,|\nabla u|^{\,p}\,\omega^p\, dx\\
&\qquad + C(1+\varepsilon+\varepsilon^{1-p})\Big(r^n(1+\ell^p) + \delta\mu(B) + \delta M^{p-1}\int_L (\sigma^p + |\nabla\sigma|^p)\, dx\\
&\qquad + \delta^p\int_E (1+v)^{\gamma}(\eta^p + |\nabla\eta|^p)\, dx\Big).\end{aligned}$$

Choosing ε small enough it follows

$$\begin{aligned}&\int_L (1+v)^{-\tau}\,|\nabla u|^{\,p}\,\omega^p\, dx\\
&\qquad\le C\Big(r^n(1+\ell^p) + \delta^p\int_E (1+v)^{\gamma}(\eta^p + |\nabla\eta|^p)\, dx\\
&\qquad\quad + \delta M^{p-1}\int_L (\sigma^p + |\nabla\sigma|^p) + \delta\mu(B)\Big).\end{aligned}\tag{3.19}$$

From (3.10) and (3.19) we get

$$\begin{aligned}\int_L |\nabla w_\delta|^p\,dx &\le C\delta^{-p}r^n(1+\ell^p) + C\int_E (1+v)^\gamma(\eta^p + |\nabla\eta|^p)\,dx\\ &\quad + C\delta^{1-p}\Big(M^{p-1}\int_L(\sigma^p + |\nabla\sigma|^p) + \mu(B)\Big).\end{aligned}\tag{3.20}$$

Since

$$\int_E (1+v)^\gamma(\eta^p + |\nabla\eta|^p)\,dx \le Cr^{-p}\int_E (1+v)^\gamma\,dx$$

and

$$\sigma^p + |\nabla\sigma|^p \le Cr^{-p}\varphi^p + |\nabla\varphi|^p + |\nabla\psi|^p,$$

it follows that

$$\begin{aligned}\int_L |\nabla w_\delta|^p\,dx &\le Cr^{-p}\int_E (1+v)^\gamma\,dx + C\delta^{-p}r^n(1+\ell)^p\\ &\quad + C\delta^{1-p}\Big(M^{p-1}\int_L (r^{-p}\varphi^p + |\nabla\varphi|^p + |\nabla\psi|^p)\,dx + \mu(B)\Big).\end{aligned}$$

This proves the part (a).
(b) We consider $\kappa > 0$; its choice will be specified latter. We continue to use the notation introduced in the course of the proof of (a) with the choice

$$\delta := \Big(\frac{1}{\kappa r^n}\int_L (u-\ell)^\gamma\omega^q\,dx\Big)^{1/\gamma}.$$

Notice that

$$\kappa = r^{-n}\int_L v^\gamma\omega^q\,dx.\tag{3.21}$$

By (3.7) and (3.21),

$$\begin{aligned}2\kappa r^n &= 2\int_L v^\gamma\omega^q\,dx\\ &\le 2^{-n}\int_L \omega^q\,dx + \int_{L\cap\{v^\gamma\ge 2^{-n-1}\}} v^\gamma\omega^q\,dx\\ &\le 2^{-n}\big(|E| + \int_F \sigma^q\,dx\big) + \int_{L\cap\{v^\gamma\ge 2^{-n-1}\}} v^\gamma\omega^q\,dx\\ &\le \kappa r^n + \int_{L\cap\{v^\gamma\ge 2^{-n-1}\}} v^\gamma\omega^q\,dx + \int_B \sigma^q\,dx\end{aligned}$$

and thus

$$\begin{aligned}\kappa r^n &\le \int_{B\cap\{v^\gamma\ge 2^{-n-1}\}\}} v^\gamma\omega^q\,dx + \int_B \sigma^q\,dx\\ &\le C\Big(\int_L w_\delta^q\,dx + \int_B \sigma^q\,dx\Big)\,.\end{aligned}$$

We apply the Sobolev inequality to the functions w_δ and σ and obtain

$$\begin{aligned}\kappa^{p/q} &\le \big(r^{-n}\int_{B\cap\Omega} w_\delta^q\,dx + r^{-n}\int_B \sigma^q\,dx\big)^{p/q}\\ &\le Cr^{p-n}\Big(\int_{B\cap\Omega}|\nabla w_\delta|^p\,dx + \int_B |\nabla\sigma|^p\,dx\Big).\end{aligned}\tag{3.22}$$

From (a) we obtain

$$\begin{aligned}r^{n-p}\kappa^{p/q} &\le C\Big(\int_L |\nabla w_\delta|^p\,dx + \int_B |\nabla\sigma|^p\,dx\Big)\\ &\le Cr^{-p}\int_E (1+v)^\gamma\,dx + C\delta^{-p}r^n(1+\ell)^p\\ &\quad + C\delta^{1-p}\Big((\delta+M)^{p-1}\int_L(\sigma^p+|\nabla\sigma|^p)\,dx + \mu(B)\Big).\end{aligned}\tag{3.23}$$

By (3.7) and (3.8),

$$\begin{aligned}\int_E (1+v)^\gamma\,dx &\le C(|E| + \int_E v^\gamma\,dx)\\ &\le C(|E| + \int_L v^\gamma\,\omega^q\,dx)\\ &\le C\kappa r^n.\end{aligned}\tag{3.24}$$

We infer from (3.23) and (3.24) that

$$\begin{aligned}\kappa^{p/q} &\le C_1\kappa + C\delta^{-p}r^p(1+\ell)^p\\ &\quad + C\delta^{1-p}r^{p-n}\Big((\delta+M)^{p-1}\int_L(r^{-p}\sigma^p+|\nabla\sigma|^p)\,dx\\ &\quad + \mu(B)\Big)\end{aligned}$$

holds for some constant C_1. If we specify κ to be so small that $\kappa^{p/q} - C_1\kappa > 0$, we obtain

$$\begin{aligned}1 &\le C\delta^{-p}r^p(1+\ell)^p\\ &\quad + C\delta^{1-p}r^{p-n}\Big((\delta+M)^{p-1}\int_L(r^{-p}\sigma^p+|\nabla\sigma|^p)\,dx + \mu(B)\Big).\end{aligned}$$

It follows that either

$$1 \le C\delta^{-p}r^p(1+\ell)^p$$

or

$$1 \le C\delta^{1-p}r^{p-n}\Big((\delta+M)^{p-1}\int_L(r^{-p}\sigma^p+|\nabla\sigma|^p)\,dx + \mu(B)\Big).$$

Anyway we deduce

$$\begin{aligned}
&\Big(\frac{1}{\kappa r^n}\int_L (u-\ell)^\gamma \psi^q \eta^q\,dx\Big)^{(p-1)/\gamma}\\
&\quad = \delta^{p-1} \le C r^{p-1}(1+\ell)^{p-1}\\
&\qquad + C r^{p-n}\Big((\delta+M)^{p-1}\int_B (r^{-p}\sigma^p + |\nabla\sigma|^p)\,dx + \mu(B)\Big).
\end{aligned}$$

Taking into account the estimates

$$r^{-p}\sigma^p + |\nabla\sigma|^p \le C(r^{-p}\varphi^p + |\nabla\varphi|^p + |\nabla\psi|^p)$$

and

$$\delta \le CM,$$

we conclude the proof. □

3.3 Theorem. *Let u be a subsolution of $-\operatorname{div}\mathbf{A} + \mathbf{B} = \mu$ in Ω. Suppose that either u is upper bounded or $b_0 = 0$. Then*

$$\begin{aligned}
&p\text{-fine-}\limsup_{x\to z} u(x) \le C\Bigg(\Big(r_0^{-n}\int_{B(x_0,r_0)\cap\Omega\cap\{u>0\}} u^\gamma\,dx\Big)^{1/\gamma}\\
&\quad + \int_0^{r_0}\Big(\frac{\mu B(x_0,r)}{r^{n-p}}\Big)^{1/(p-1)}\frac{dr}{r} \qquad (3.25)\\
&\quad + (1+\|u\|_\infty)\int_0^{2r_0}\Big(\frac{\operatorname{cap}_p(B(x_0,r)\setminus\Omega,\, r)}{r^{n-p}}\Big)^{1/(p-1)}\frac{dr}{r}\Bigg)
\end{aligned}$$

for all $x_0 \in \overline{\Omega}$ and $r_0 \le R_0$.

Proof: We denote $M = 1 + \|u\|_\infty$ and set $\kappa \in (0,1)$ to be the constant from Lemma 3.2. We set $r_j = 2^{-j}r_0$ and pick cutoff functions η_j such that $0 \le \eta_j \le 1$, $\eta_j = 0$ outside $B(x_0, r_j)$, $\eta_j = 1$ on $B(x_0, r_{j+1})$ and $|\nabla\eta_j| \le 5/r_j$. Further, we find functions $g_j \in W^{1,p}(\mathbf{R}^n)$ such that $0 \le g_j \le 1$, the interior of $\{g_j = 1\}$ contains $B(x_0, r_j)\setminus\Omega$ and

$$\int_{\mathbf{R}^n} (r_j^{-p} g_j^p + |\nabla g_j|^p)\,dx \le C \operatorname{cap}_p(B(x_0,r_j)\setminus\Omega,\, r_j)\,. \tag{3.26}$$

We denote

$$\begin{aligned}
\psi_j &= \min(1,\,(2-3g_j)^+),\\
\varphi_j &= \min(1,\,3g_j + 3g_{j-1}), \quad j \ge 1,\\
B_j &= B(x_0, r_j),\\
L_j &= B_j \cap \Omega \cap \{u \ge \ell_j\}\\
E_j &= L_j \cap \{\varphi_j < 1\},\\
F_j &= L_j \cap \{\varphi_j = 1\}.
\end{aligned}$$

Then by (3.26),

$$\begin{aligned}\int_{B_j} \bigl(r_j^{-p}\varphi_j^p + |\nabla\varphi|_j^p\bigr)\,dx &\le C\,\operatorname{cap}_p(B_{j-1}\setminus\Omega,\,r_j),\\ \int_{B_j} |\nabla\psi|_j^p\,dx &\le C\,\operatorname{cap}_p(B_j\setminus\Omega,\,r_j).\end{aligned} \tag{3.27}$$

We define recursively $\ell_0 = 0$,

$$\ell_{j+1} = \ell_j + \Bigl(\frac{1}{\kappa r_j^n}\int_{L_j}(u-\ell_j)^\gamma\psi_j^q\eta_j^q\,dx\Bigr)^{1/\gamma}, \qquad j = 0,1,2,\dots$$

We write

$$\delta_j = \ell_{j+1} - \ell_j.$$

We claim that, for $j \ge 1$,

$$\begin{aligned}\delta_j \le \frac{1}{2}\,\delta_{j-1} + C\Biggl(&r_j(1+\ell_j) + \Bigl(\frac{\mu B_j}{r_j^{n-p}}\Bigr)^{1/(p-1)}\\ &+ M\Bigl(\frac{\operatorname{cap}_p(B_{j-1}\setminus\Omega,\,r_j)}{r_j^{n-p}}\Bigr)^{1/(p-1)}\Biggr).\end{aligned} \tag{3.28}$$

This is trivial when $\delta_j \le \frac{1}{2}\,\delta_{j-1}$, so assume that $\delta_{j-1} \le 2\delta_j$. In this case, since $\psi_{j-1}\eta_{j-1} = 1$ on E_j, we have

$$\begin{aligned}|E_j| &\le \delta_{j-1}^{-\gamma}\int_{L_{j-1}}(u-\ell_{j-1})^\gamma\,\psi_{j-1}\eta_{j-1}\,dx\\ &= \kappa r_{j-1}^n \le 2^n\kappa r_j^n\end{aligned} \tag{3.29}$$

and

$$\begin{aligned}&\int_{E_j}(u-\ell_j)^\gamma\,dx\\ &\le \int_{L_{j-1}}(u-\ell_{j-1})^\gamma\psi_{j-1}^q\eta_{j-1}^q\,dx = \delta_{j-1}^\gamma\kappa r_{j-1}^n = 2^{n+\gamma}\delta_j^\gamma\kappa r_j^n\\ &= 2^{n+\gamma}\int_{L_j}(u-\ell_j)^\gamma\psi_j^q\eta_j^q\,dx.\end{aligned} \tag{3.30}$$

Thus (3.7) and (3.8) are verified and Lemma 3.2 yields

$$\begin{aligned}\delta_j \le C\Biggl(&r_j(1+\ell_j) + \Bigl(\frac{\mu B_j}{r_j^{n-p}}\Bigr)^{1/(p-1)}\\ &+ M\Bigl(\frac{\operatorname{cap}_p(B_{j-1}\setminus\Omega,\,r_j)}{r_j^{n-p}}\Bigr)^{1/(p-1)}\Biggr)\end{aligned}$$

which proves (3.28). Summing up (3.28) for $j = 1, \dots, k$ we get

$$\begin{aligned}
\frac{1}{2}\ell_{k+1} &= \frac{1}{2}(\delta_0 + \cdots + \delta_k) \le \delta_k + \frac{1}{2}(\delta_0 + \cdots + \delta_{k-1}) \\
&\le \delta_0 + C\Bigg(\sum_{j=1}^{k} r_j(1+\ell_{j+1}) + \sum_{j=1}^{k}\Big(\frac{\mu B_j}{r_j^{n-p}}\Big)^{1/(p-1)} \\
&\quad + M\sum_{j=1}^{k}\Big(\frac{\operatorname{cap}_p(B_{j-1}\setminus\Omega, r)}{r_j^{n-p}}\Big)^{1/(p-1)}\Bigg) \\
&\le C\, r_0\ell_{k+1} + C\Bigg(\Big(r_0^{-n}\int_{E_0} u^\gamma\,dx\Big)^{1/\gamma} \\
&\quad + \sum_{j=1}^{k}\int_{r_j}^{r_{j-1}}\Big(\frac{\mu B(x_0,r)}{r^{n-p}}\Big)^{1/(p-1)}\,\frac{dr}{r} \\
&\quad + M\sum_{j=1}^{k}\int_{r_j}^{r_{j-1}}\Big(\frac{\operatorname{cap}_p(B(x_0,2r)\setminus\Omega, r)}{r^{n-p}}\Big)^{1/(p-1)}\,\frac{dr}{r}\Bigg).
\end{aligned}$$

If $r_0 \le R_1 := \frac{1}{2C_2}$, we obtain

$$\begin{aligned}
\lim_j \ell_j \le C\Bigg(&\Big(r_0^{-n}\int_{B(x_0,r_0)\cap\Omega\cap\{u>0\}} u^\gamma\,dx\Big)^{1/\gamma} \\
&+ \int_0^{r_0}\Big(\frac{\mu B(x_0,r)}{r^{n-p}}\Big)^{1/(p-1)}\,\frac{dr}{r} \\
&+ M\int_0^{2r_0}\Big(\frac{\operatorname{cap}_p(B(x_0,r)\setminus\Omega, r)}{r^{n-p}}\Big)^{1/(p-1)}\,\frac{dr}{r}\Bigg).
\end{aligned} \tag{3.31}$$

If $R_1 < r_0 < R_0$, then (3.31) holds as well, because then

$$r_0/R_1 \le R_0/R_1 \le C.$$

It remains to prove that

$$p\text{-fine-}\limsup_{x\to z} u(x) \le \lim_j \ell_j\,. \tag{3.32}$$

We may assume that the right hand part of (3.25) is finite, otherwise the assertion of the theorem is trivial. We choose $\varepsilon > 0$ and denote $\ell = \lim_j \ell_j$. Set

$$w_j = (2^{\gamma/q}-1)^{-1}\Big(\Big(1+\frac{(u-\ell-\varepsilon)^+}{\varepsilon}\Big)^{\gamma/q} - 1\Big)\psi_j\eta_j$$

on Ω and $w_j = 0$ elsewhere. Then $w_j \in W_0^{1,p}(B_j)$, $w_j + \varphi_j\eta_j \ge 1$ on $B_{j+1} \cap \Omega \cap \{u > \ell + 2\varepsilon\}$ and thus

$$\mathrm{cap}_p(B_{j+1} \cap \Omega \cap \{u > \ell + 2\varepsilon\}, r_j) \le C \int_{B_j} (|\nabla w_j|^p + |\nabla(\varphi_j\eta_j)|^p)\,dx.$$

Denote

$$E_j' = B_j \cap \Omega \cap \{u > \ell + \varepsilon\} \cap \{\varphi_j < 1\}.$$

Using Lemma 3.2.a we obtain

$$\begin{aligned}
&\mathrm{cap}_p(B_{j+1} \cap \Omega \cap \{u > \ell + 2\varepsilon\}, r_j)\\
&\quad\le C \int_{B_j} (|\nabla w_j|^p + |\nabla(\varphi_j\eta_j)|^p)\,dx \le C\Big(\varepsilon^{-p} r_j^n (1 + (\ell+\varepsilon)^p)\\
&\qquad + r_j^{-p} \int_{E_j'} \Big(1 + \frac{u - \ell - \varepsilon}{\varepsilon}\Big)^\gamma\,dx\\
&\qquad + \varepsilon^{1-p}\,\mu(B_j)\\
&\qquad + (1 + \varepsilon^{1-p})(1 + \|u\|_\infty)^{p-1} \int_{B_j} (r_j^{-p}\varphi_j^p + |\nabla\varphi_j|^p + |\nabla\psi_j|^p)\,dx\Big).
\end{aligned}$$

It follows

$$\begin{aligned}
&\sum_j \Big(\frac{\mathrm{cap}_p(B_{j+1} \cap \Omega \cap \{u > \ell + 2\varepsilon\}, r_j)}{r_j^{n-p}}\Big)^{1/(p-1)}\\
&\quad\le C\Bigg(\varepsilon^{-p/(p-1)} r_0^{p/(p-1)} (1 + \ell^p)^{1/(p-1)}\\
&\qquad + \sum_j \Big(r_j^{-n} \int_{E_j'} \Big(1 + \frac{u - \ell - \varepsilon}{\varepsilon}\Big)^\gamma\,dx\Big)^{1/(p-1)}\\
&\qquad + \varepsilon^{-1} \sum_j \Big(\frac{\mu(B_j)}{r_j^{n-p}}\Big)^{1/(p-1)}\\
&\qquad + (1 + \varepsilon^{-1})(1 + \|u\|_\infty) \sum_j \Big(\frac{\mathrm{cap}_p(B_j \setminus \Omega, r_j)}{r_j^{n-p}}\Big)^{1/(p-1)}\Bigg).
\end{aligned}\tag{3.33}$$

Note that

$$\sum_{j=0}^\infty (\delta_j/\ell)^{\gamma/(p-1)} \le \sum_{j=0}^\infty (\delta_j/\ell) = 1.\tag{3.34}$$

Using (3.27) and (3.34) we estimate

$$\begin{aligned}
&\sum_j \Big(r_j^{-n} \int_{E_j'} \Big(1 + \frac{u-\ell-\varepsilon}{\varepsilon}\Big)^\gamma dx\Big)^{1/(p-1)} \\
&\quad \le C \sum_j \Big(r_j^{-n} \int_{E_j} \varepsilon^{-\gamma}(u-\ell_{j-1})^\gamma \, dx\Big)^{1/(p-1)} \\
&\quad \le C \sum_j \Big(r_j^{-n} \int_{L_{j-1}} \varepsilon^{-\gamma}(u-\ell_{j-1})^\gamma \psi_{j-1}\varphi_{j-1} \, dx\Big)^{1/(p-1)} \\
&\quad \le C \sum_j (\kappa\varepsilon^{-\gamma}\delta_{j-1}^\gamma)^{1/(p-1)} < \infty.
\end{aligned}$$

If the right hand part of (3.32) is finite, then the remaining sums on the right hand part of (3.33) also converge (we assumed this), so that the set

$$\Omega \cap \{u > \ell + 2\varepsilon\}$$

is p-thin at x_0 for any $\varepsilon > 0$. We proved (3.32), which concludes the proof. □

4. Necessity of the Wiener condition

4.1 Example. Let Ω be a bounded open set and let $u_0 \in W^{1,p}(\Omega)$. Consider the Dirichlet problem

$$\begin{cases} -\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = 0, \\ u - u_0 \in W_0^{1,p}(\Omega). \end{cases} \tag{4.1}$$

Then we obtain a unique solution u of (4.1) by minimizing

$$\int_\Omega |\nabla v|^p \, dx$$

in the closed convex set

$$\{v \in W^{1,p}(\Omega)\colon v - u_0 \in W_0^{1,p}(\Omega)\}.$$

Since

$$\int_\Omega |\nabla u|^p \, dx \le \int_\Omega |\nabla u_0|^p \, dx,$$

using Poincaré's inequality we get

$$\begin{aligned}
\int_\Omega |u|^p \, dx &\le C\Big(\int_\Omega |u_0|^p \, dx + \int_\Omega |u-u_0|^p \, dx\Big) \\
&\le C\Big(\int_\Omega |u_0|^p \, dx + \int_\Omega |\nabla u - \nabla u_0|^p \, dx\Big) \\
&\le C\Big(\int_\Omega |u_0|^p \, dx + \int_\Omega |\nabla u|^p + |\nabla u_0|^p \, dx\Big) \\
&\le C\Big(\int_\Omega |u_0|^p \, dx + \int_\Omega |\nabla u_0|^p \, dx\Big).
\end{aligned}$$

Let $M = \|u_0\|_\infty < \infty$. If we test the minimizing property by

$$v(x) = \begin{cases} u, & |u| \le M, \\ M, & u > M, \\ -M, & u < M, \end{cases}$$

then we get that $u \le M$ a.e. Similar estimates hold for all equations of the monotone type.

4.2 Theorem. *In addition to* (2.1), *suppose that for any $u_0 \in \mathcal{C}_c^1(\mathbf{R}^n)$ there is $u \in W^{1,p}(\Omega)$ such that*

$$\begin{cases} -\operatorname{div} \mathbf{A} + \mathbf{B} = 0, \\ u - u_0 \in W_0^{1,p}(\Omega), \end{cases} \tag{4.2}$$

and

$$\int_\Omega |u|^p\,dx \le C \int_\Omega \big(|u_0|^p + |\nabla u_0|^p\big)\,dx,\ \ \|u\|_\infty \le C\|u_0\|_\infty \tag{4.3}$$

with a constant C independent of u_0. Let $z \in \partial\Omega$ and suppose that

$$\mathbf{w}_p(z, \mathbf{R}^n \setminus \Omega) < \infty.$$

Then z is irregular for the equation

$$-\operatorname{div} \mathbf{A} + \mathbf{B} = 0.$$

Proof: Choose $\varepsilon > 0, \rho \in (0,1)$ to be specified later. The singleton $\{z\}$ has zero p-capacity. Hence, we find a $\mathcal{C}^1$-function u_0 on $\mathbf{R}^n$ supported in $B(z,1)$ such that $u_0(z) = 1$ and $\int_{\mathbf{R}^n} \big(|u_0|^p + |\nabla u_0|^p\big)\,dx < \varepsilon$. Let u be a continuous solution of (4.2), (4.3). By Theorem 3.3,

$$\begin{aligned} p\text{-fine-}\limsup_{x\to z} u(x) &\le C_1\Big(\rho^{-n} \int_{B(z,\rho)} u^\gamma\,dx\Big)^{1/\gamma} \\ &\quad + C_2 \int_0^\rho \Big(\frac{\operatorname{cap}_p(B(z,r)\setminus\Omega)}{r^{n-p}}\Big)^{1/(p-1)} \frac{dr}{r}. \end{aligned}$$

Hölder's inequality yields

$$\Big(\rho^{-n} \int_{B(x,\rho)} |u^\gamma|\,dx\Big)^{1/\gamma} \le C\rho^{-n/p}\Big(\int_{B(x,\rho)} |u|^p\,dx\Big)^{1/p} \le C_3 \rho^{-n/p}\varepsilon^{1/p}.$$

Since $\mathbf{R}^n \setminus \Omega$ is p-thin at z, we can find $\rho \in (0,1)$ such that

$$C_2 \int_0^\rho \Big(\frac{\operatorname{cap}_p(B(z,r)\setminus\Omega)}{r^{n-p}}\Big)^{1/(p-1)} \frac{dr}{r} < \frac{1}{3}.$$

Then we can specify the choice of ε so that

$$C_1 C_3 \rho^{-n/p}\varepsilon^{1/p} \le \frac{1}{3}.$$

We obtain that

$$p\text{-fine-}\limsup_{x\to z} u(x) < 1 = u_0(z),$$

hence z is not regular. □

Note added in proof. In a new preprint Gianazza, Marchi and Villani prove Wiener criteria for a related class of equations which is neither a subclass, nor a superclass of the class of equations investigated here.

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