Sequential closures of σ-subalgebras for a vector measure

W.J. Ricker

Abstract. Let X be a locally convex space, $m : \Sigma \to X$ be a vector measure defined on a σ-algebra Σ, and $L^1(m)$ be the associated (locally convex) space of m-integrable functions. Let $\Sigma(m)$ denote $\{\chi_E; E \in \Sigma\}$, equipped with the relative topology from $L^1(m)$. For a subalgebra $\mathcal{A} \subseteq \Sigma$, let $\mathcal{A}_\sigma$ denote the generated σ-algebra and $\overline{\mathcal{A}}_s$ denote the *sequential* closure of $\chi(\mathcal{A}) = \{\chi_E; E \in \mathcal{A}\}$ in $L^1(m)$. Sets of the form $\overline{\mathcal{A}}_s$ arise in criteria determining separability of $L^1(m)$; see [6]. We consider some natural questions concerning $\overline{\mathcal{A}}_s$ and, in particular, its relation to $\chi(\mathcal{A}_\sigma)$. It is shown that $\overline{\mathcal{A}}_s \subseteq \Sigma(m)$ and moreover, that $\{E \in \Sigma; \chi_E \in \overline{\mathcal{A}}_s\}$ is always a σ-algebra and contains $\mathcal{A}_\sigma$. Some properties of X are determined which ensure that $\chi(\mathcal{A}_\sigma) = \overline{\mathcal{A}}_s$, for any X-valued measure m and subalgebra $\mathcal{A} \subseteq \Sigma$; the class of such spaces X turns out to be quite extensive.

Keywords: σ-subalgebra, vector measure, sequential closure
Classification: 28B05

Let X be a locally convex Hausdorff space (briefly, lcHs), Σ be a σ-algebra of subsets of some set Ω and $m : \Sigma \to X$ be a *vector measure* (i.e. m is σ-additive). Associated with m is a lcHs $L^1(m)$ of m-integrable functions. Just as for scalar measures, an important property is the separability of $L^1(m)$; see [6]. In particular, if $\Sigma(m)$ denotes the subset $\{\chi_E; E \in \Sigma\}$ of $L^1(m)$, then one criteria which ensures the separability of $L^1(m)$ is the existence of a countably generated σ-algebra $\Sigma_0 \subseteq \Sigma$ such that $\Sigma(m) = \Sigma_0(m)$, [6, Proposition 2]. So, the idea is to look for algebras of sets $\mathcal{A} \subseteq \Sigma$, hopefully countable, such that the generated σ-algebra $\mathcal{A}_\sigma$ satisfies $\mathcal{A}_\sigma(m) = \Sigma(m)$. A closely related set is the sequential closure, $\overline{\mathcal{A}}_s$, of the set $\chi(\mathcal{A}) = \{\chi_E; E \in \mathcal{A}\}$, formed in the topological space $L^1(m)$. It is always the case that $\chi(\mathcal{A}_\sigma) \subseteq \overline{\mathcal{A}}_s$ and, if the range, $m(\Sigma) = \{m(E); E \in \Sigma\}$, of m is metrizable for the relative topology from X, then actually $\overline{\mathcal{A}}_s \subseteq \Sigma(m)$ and $\chi(\mathcal{A}_\sigma) = \overline{\mathcal{A}}_s$, [6, Proposition 3].

The purpose of this note is to consider the following questions.

(A) Is it always the case that $\overline{\mathcal{A}}_s$ is a sequentially closed subset of $\Sigma(m)$, rather than just of $L^1(m)$?

(B) Is $\{E; \chi_E \in \overline{\mathcal{A}}_s\}$ actually a σ-algebra and is it contained in Σ?

(C) Is it always the case that $\chi(\mathcal{A}_\sigma) = \overline{\mathcal{A}}_s$?

The first question was raised in [6, Remark 5 (i)]. It will be shown that Questions A & B have an affirmative answer. The final section is concerned with

Question C. By the remarks above $\chi(\mathcal{A}_\sigma) = \overline{\mathcal{A}}_s$ whenever X is a Fréchet lcHs or has the property that bounded sets are metrizable (e.g. the strict inductive limit of a sequence of Fréchet spaces). It will be shown that Question C has a positive answer in a much larger class of lcH-spaces.

1. Preliminaries

Let X be a lcHs and $m : \Sigma \to X$ be a vector measure. A Σ-measurable function $f : \Omega \to \mathbb{C}$ is called *m-integrable* if it is integrable with respect to the complex measure $\langle m, x' \rangle : E \mapsto \langle m(E), x' \rangle$, for $E \in \Sigma$, for every $x' \in X'$ (the continuous dual space of X) and if, for every $E \in \Sigma$, there exists an element of X, denoted by $\int_E f dm$, which satisfies $\langle \int_E f dm, x' \rangle = \int_E f d\langle m, x' \rangle$, for every $x' \in X'$. The linear space of all m-integrable functions is denoted by $L(m)$. Let $\mathcal{P}(X)$ denote the family of all continuous seminorms in X or, at least enough seminorms to determine the given lc-topology τ in X. Each $q \in \mathcal{P}(X)$ induces a seminorm $q(m)$ in $L(m)$ via the formula

$$(1) \qquad q(m) : f \mapsto \sup\{\int_\Omega |f| d|\langle m, x' \rangle|; x' \in U_q^0\}, \qquad f \in L(m),$$

where $|\nu|$ denotes the total variation measure of a complex measure $\nu : \Sigma \to \mathbb{C}$ and $U_q^0 \subseteq X'$ denotes the polar of the closed q-unit ball $U_q = q^{-1}([0,1])$. The seminorms (1), as q varies through $\mathcal{P}(X)$, define a lc-topology $\tau(m)$ in $L(m)$. Since $\tau(m)$ may not be Hausdorff we form the usual quotient space of $L(m)$ with respect to the closed subspace $\cap_{q \in \mathcal{P}(X)} q(m)^{-1}(\{0\})$. The resulting Hausdorff space (with topology again denoted by $\tau(m)$) is denoted by $L^1(m)$; it can be identified with equivalence classes of functions from $L(m)$ modulo m-null functions, where a function $f \in L(m)$ is *m-null* whenever $\int_E f dm = 0$, for every $E \in \Sigma$. All of the above definitions and further properties of $L^1(m)$ can be found in [4].

Let $\Sigma(m)$ denote the subset of $L^1(m)$ corresponding to $\{\chi_E; E \in \Sigma\} \subseteq L(m)$. Elements of $\Sigma(m)$ will be identified with equivalence classes of elements from Σ. The topology $\tau(m)$ of $L^1(m)$ induces a topology on $\Sigma(m)$ by restriction (again denoted by $\tau(m)$).

Let Λ be a topological Hausdorff space and $Y \subseteq \Lambda$. Then $[Y]$ denotes the set of all elements in Λ which are the limit of some sequence of points from Y. A set $Y \subseteq \Lambda$ is called *sequentially closed* if $Y = [Y]$. The *sequential closure* $\overline{Y}_s$, of a set $Y \subseteq \Lambda$, is the smallest sequentially closed subset of Λ which contains Y. Alternatively, let $Y_0 = Y$. Let Ω_1 be the smallest uncountable ordinal. Suppose that $0 < \alpha < \Omega_1$ and that Y_β has been defined for all ordinals β satisfying $0 \le \beta < \alpha$. Define $Y_\alpha = [\cup_{0 \le \beta < \alpha} Y_\beta]$. Then $\overline{Y}_s = \cup_{0 \le \alpha < \Omega_1} Y_\alpha$.

2. Questions A and B

Throughout this section X is a lcHs. Given a vector measure $m : \Sigma \to X$ and a $\mathbb{R}$-valued function $f \in L(m)$ we define $A(f) = \{w \in \Omega; |1 - f(w)| \le \frac{1}{2}\}$.

Lemma 1. *Let $f \in L^1(m)$ be $\mathbb{R}$-valued. Then, for every $E \in \Sigma$,*

$$|\chi_E - \chi_{A(f)}| \leq 2|\chi_E - f|.$$

PROOF: follows from the identity $|\chi_E - \chi_F| = \chi_{E \triangle F}$, valid for every $E, F \in \Sigma$, where $E \triangle F = (E \backslash F) \cup (F \backslash E)$. □

Proposition 1. *Let $m : \Sigma \to X$ be a vector measure. Then $\Sigma(m)$ is a $\tau(m)$-closed subset of $L^1(m)$.*

PROOF: Given any $f \in L^1(m)$ and $E \in \Sigma$, Lemma 1 implies that

$$|\chi_E - \chi_{A(Re(f))}| \leq 2|\chi_E - Re(f)| = 2|Re(\chi_E - f)| \leq 2|\chi_E - f|.$$

These inequalities and (1) show that

$$q(m)(\chi_E - \chi_{A(Re(f))}) \leq 2q(m)(\chi_E - f), \qquad q \in \mathcal{P}(X).$$

It follows that if $\{\chi_{E(\alpha)}\}$ is a net in $\Sigma(m)$ which is $\tau(m)$-convergent to $f \in L^1(m)$, then $f = \chi_{A(Re(f))}$ and so $f \in \Sigma(m)$. □

Remark 1. (i) An affirmative answer to Question A is now immediate from Proposition 1 and the fact that $\chi(\mathcal{A}) \subseteq \Sigma(m)$ with $\overline{\mathcal{A}}_s$ being the sequential closure of $\chi(\mathcal{A})$ in $L^1(m)$.

(ii) For the particular case of $\mathcal{A} = \Sigma$, Proposition 1 implies that $\overline{\mathcal{A}}_s = \Sigma(m)$ is not just sequentially closed in $L^1(m)$ but, is actually closed. This is *not* typically the case for a *proper* σ-subalgebra $\mathcal{A} \subsetneq \Sigma$. For instance, let $X = \mathbb{C}^{[0,1]}$ denote the vector space of all $\mathbb{C}$-valued functions on $\Omega = [0, 1]$ equipped with pointwise operations. Then X is a (complete) lcHs for the topology τ of pointwise convergence on Ω. Let Σ denote the σ-algebra of all subsets of Ω and define a vector measure $m : \Sigma \to X$ by $m(E) = \chi_E$, for $E \in \Sigma$. It turns out that every function $f : \Omega \to \mathbb{C}$ belongs to $L^1(m)$ and $\int_E f dm = \chi_E f$, for $E \in \Sigma$. The topology $\tau(m)$ is the topology in $L^1(m)$ of pointwise convergence on Ω. Let $\mathcal{A} \subset \Sigma$ be the σ-algebra of all Borel sets. Then $\overline{\mathcal{A}}_s = \chi(\mathcal{A})$ which is clearly sequentially closed in $L^1(m)$ but, is surely not closed. □

The answer to Question B is provided by the following

Proposition 2. *Let $m : \Sigma \to X$ be a vector measure and $\mathcal{A} \subseteq \Sigma$ be an algebra of sets. Then $\{E; \chi_E \in \overline{\mathcal{A}}_s\}$ is a σ-subalgebra of Σ and contains $\mathcal{A}_\sigma$.*

PROOF: Define $\mathcal{A}_0 = \chi(\mathcal{A}) \subseteq \Sigma(m)$ and $\mathcal{A}_1 = [\mathcal{A}_0]$. Let $\chi_E \in \mathcal{A}_1$, say $\chi_E = \lim \chi_{E(n)}$ where $E(n) \in \mathcal{A}$ for $n = 1, 2, \cdots$. Since $\chi_E - \chi_{E(n)} = \chi_{E(n)^c} - \chi_{E^c}$, for all $n = 1, 2, \ldots$, it follows from (1) that

$$q(m)(\chi_{E^c} - \chi_{E(n)^c}) = q(m)(\chi_E - \chi_{E(n)}), \qquad q \in \mathcal{P}(X).$$

Accordingly, $\chi_{E(n)^c} \to \chi_{E^c}$ in $\Sigma(m)$. Hence, $\chi_{E^c} \in \mathcal{A}_1$ whenever $\chi_E \in \mathcal{A}_1$.

Suppose also that $\chi_F \in \mathcal{A}_1$ and $F(n) \in \mathcal{A}$, for $n = 1, 2, \cdots$, are sets such that $\chi_{F(n)} \to \chi_F$ in $\Sigma(m)$. Since $\mathcal{A}$ is an algebra $F(n) \cap E(n) \in \mathcal{A}$, for each $n = 1, 2, \ldots$. Moreover,

$$|\chi_{E\cap F} - \chi_{E(n)\cap F(n)}| \leq |\chi_E - \chi_{E(n)}|\chi_F + |\chi_F - \chi_{F(n)}|\chi_{E(n)}$$

and hence, for each $q \in \mathcal{P}(X)$,

$$q(m)(\chi_{E\cap F} - \chi_{E(n)\cap F(n)}) \leq q(m)((\chi_E - \chi_{E(n)})\chi_F) + q(m)((\chi_F - \chi_{F(n)})\chi_{E(n)}).$$

But, it is clear from (1) that $q(m)(\chi_R f) \leq q(m)(f)$, for every $R \in \Sigma$ and $f \in L^1(m)$, from which it follows that

$$q(m)(\chi_{E\cap F} - \chi_{E(n)\cap F(n)}) \leq q(m)(\chi_E - \chi_{E(n)}) + q(m)(\chi_F - \chi_{F(n)}).$$

Accordingly, also $\chi_{E\cap F} \in \mathcal{A}_1$ whenever $\chi_E, \chi_F \in \mathcal{A}_1$. Hence, $\{E; \chi_E \in \mathcal{A}_1\}$ is an algebra of subsets of Σ.

By a transfinite induction argument it now follows that $\{E; \chi_E \in \overline{\mathcal{A}}_s\} = \cup_{0\leq\alpha<\Omega_1}\{E; \chi_E \in \mathcal{A}_\alpha\}$ is an increasing union of algebras of sets from Σ and hence, is itself an algebra of sets from Σ.

Suppose that $\{E(n)\}_{n=1}^{\infty}$ is a monotone sequence from $\{E; \chi_E \in \overline{\mathcal{A}}_s\}$ with limit $E \in \Sigma$, say. Then $\{\chi_{E(n)}\}_{n=1}^{\infty}$ is a sequence in $\Sigma(m)$ with pointwise limit χ_E. Let $j : X \to \widehat{X}$ be an isomorphism of X onto a dense subspace $j(X)$ of its completion $\widehat{X}$. Then the set function $\hat{m} : \Sigma \to \widehat{X}$ given by $\hat{m} = j \circ m$ is a vector measure and $L^1(m)$ is a linear subspace of $L^1(\hat{m})$. Moreover, each $q \in \mathcal{P}(X)$ has a unique extension to a continuous seminorm $\hat{q} \in \mathcal{P}(\widehat{X})$ which satisfies $\hat{q}(\hat{m})(\chi_F) = q(m)(\chi_F)$, for every $F \in \Sigma$. Accordingly,

$$q(m)(\chi_E - \chi_{E(n)}) = q(m)(\chi_{E\triangle E(n)}) = \hat{q}(\hat{m})(\chi_{E\triangle E(n)}) = \hat{q}(\hat{m})(\chi_E - \chi_{E(n)}),$$

for each $n = 1, 2, \ldots$. By the Dominated Convergence Theorem for vector measures in sequentially complete spaces, [4, II Theorem 4.2], applied to $\hat{m}$ in $\widehat{X}$, it follows that $\hat{q}(\hat{m})(\chi_E - \chi_{E(n)}) \to 0$, as $n \to \infty$, and hence, also $q(m)(\chi_E - \chi_{E(n)}) \to 0$. This shows that $\chi_{E(n)} \to \chi_E$ in $L^1(m)$. The sequential closedness of $\overline{\mathcal{A}}_s$ implies that $\chi_E \in \overline{\mathcal{A}}_s$. This shows that $\{E; \chi_E \in \overline{\mathcal{A}}_s\}$, in addition to being an algebra of sets, is also a monotone class and hence, is actually a σ-algebra.

The inclusion $\chi(\mathcal{A}_\sigma) \subseteq \overline{\mathcal{A}}_s$ is established in [6, Lemma 2 (iii)] for the case when X is sequentially complete. By passing to the completion $\widehat{X}$ and arguing as above, the proof given in [6, Lemma 2 (iii)] can easily be modified to apply in any lcHs X. $\square$

We give a simple application of Proposition 2. Let Y be a Banach space and $X = L(Y)$ be the space of all bounded linear operators from Y into itself, equipped with the strong operator topology. The notion of a Boolean algebra (briefly, B.a.) of projections which is σ-complete (in the sense of W. Bade) is by now standard, [2, Chapter XVII, §3]. This is a generalization to Banach spaces of the classical notion of the resolution of the identity of a normal operator in Hilbert space.

Corollary 2.1. *Let Y be a Banach space, $\mathcal{M} \subseteq L(Y)$ be a Bade σ-complete B.a. and $\mathcal{B} \subseteq \mathcal{M}$ be a Boolean subalgebra. Then the sequential closure $\overline{\mathcal{B}}_s$, of $\mathcal{B}$, in the lcHs $L(Y)$ is a sequentially complete, Bade σ-complete B.a. containing $\mathcal{B}$ and is minimal with respect to these properties.*

Proof: An argument along the lines of the proof of Proposition 2 shows that $\overline{\mathcal{B}}_s = \cup_{0 \le \alpha < \Omega_1} \mathcal{B}_\alpha$ is the increasing union of a family of B.a.'s and hence, is itself a B.a. It then follows from a standard result about monotone limits of sequences in a Bade σ-complete B.a., [2, XVII Lemma 3.4], that $\overline{\mathcal{B}}_s$ is Bade σ-complete. Since closed, bounded subsets of the quasicomplete lcHs $L(Y)$ are complete and $\overline{\mathcal{B}}_s$ is sequentially closed, it follows that $\overline{\mathcal{B}}_s$ is sequentially complete. The minimality condition is routine to verify. $\square$

A Bade σ-complete B.a. is a complete subset of $L(Y)$ iff it is Bade complete as a B.a., [2, XVII Corollary 3.7 & Lemma 3.23]. Hence, Corollary 2.1 is of some interest since, in applications, sequential completeness often suffices. Moreover, the sequential closure is sometimes easier to determine than the full closure in $L(Y)$.

3. Question C

Let $m : \Sigma \to X$ be a vector measure and $\mathcal{A} \subseteq \Sigma$ be an algebra of sets. Recall that $\mathcal{A}_\sigma$ is the σ-algebra generated by $\mathcal{A}$. It has been shown that always $\chi(\mathcal{A}_\sigma) \subseteq \overline{\mathcal{A}}_s$ and, under certain conditions on X (e.g. bounded sets are metrizable), it is known this inclusion is an equality. The question is whether it is always true that $\chi(\mathcal{A}_\sigma) = \overline{\mathcal{A}}_s$. Of course, this is equivalent to the question of whether $\chi(\mathcal{A}_\sigma)$ is sequentially closed in $\Sigma(m)$? The construction of $\mathcal{A}_\sigma$ from $\mathcal{A}$ is a transfinite procedure of a set theoretic nature whereas the construction of $\overline{\mathcal{A}}_s = \overline{\chi(\mathcal{A})_s}$ is a transfinite procedure of a topological nature; it is unclear whether these different processes lead to the "same" set.

It is now necessary to have a more precise notation. If we wish to indicate the dependence of the sequential closure of a subset Y of a topological space Λ on the particular topology τ under consideration, then we will denote the sequential closure by $\overline{Y_s(\tau)}$. Let X be a lcHs and $m : \Sigma \to X$ be a vector measure. Let ρ be any lcH-topology in X consistent with the duality $\langle X, X' \rangle$; for brevity we will simply call ρ a *consistent lcH-topology.* If X_ρ denotes X equipped with the topology ρ and $m_\rho : \Sigma \to X_\rho$ denotes the set function m considered as taking its values in X_ρ, then the Orlicz-Pettis theorem, [4, I Theorem 1.3], guarantees that m_ρ is also a vector measure. Clearly $L^1(m)$ and $L^1(m_\rho)$ coincide as vector spaces and $\Sigma(m)$ and $\Sigma(m_\rho)$ coincide as sets. Proposition 2 applied to m_ρ in X_ρ shows that $\chi(\mathcal{A}_\sigma) \subseteq \overline{\mathcal{A}_s(\rho)}$ for *every* consistent lcH-topology ρ. If ρ_1 is weaker than ρ_2, then clearly $\overline{\mathcal{A}_s(\rho_2)} \subseteq \overline{\mathcal{A}_s(\rho_1)}$. It follows that if $\chi(\mathcal{A}_\sigma) = \overline{\mathcal{A}_s(\rho)}$ for some consistent lcH-topology ρ, then actually $\chi(\mathcal{A}_\sigma) = \overline{\mathcal{A}_s(\nu)}$ for every consistent lcH-topology ν in X satisfying $\rho \subseteq \nu \subseteq \mu$, where μ is the Mackey topology in X. We summarise these comments in the following

Lemma 2. *Let $m : \Sigma \to X$ be a vector measure and $\mathcal{A} \subseteq \Sigma$ be an algebra of sets. If ρ is any consistent lcH-topology in X for which $\chi(\mathcal{A}_\sigma) = \overline{\mathcal{A}_s(\rho)}$, then also $\chi(\mathcal{A}_\sigma) = \overline{\mathcal{A}_s(\nu)}$ for every consistent lcH-topology ν in X satisfying $\rho \subseteq \nu \subseteq \mu$.*

The weak topology $\sigma(X, X')$ is also denoted simply by σ.

Proposition 3. *Let X be a quasicomplete lcHs with the property that its weakly compact sets are metrizable for $\sigma(X, X')$. Let $m : \Sigma \to X$ be a vector measure and $\mathcal{A} \subseteq \Sigma$ be an algebra of sets. Then $\chi(\mathcal{A}_\sigma) = \overline{\mathcal{A}_s(\rho)}$ for every consistent lcH-topology ρ in X. In particular, $\chi(\mathcal{A}_\sigma) = \overline{\mathcal{A}}_s$ where $\overline{\mathcal{A}}_s$ is formed with respect to the given topology in X.*

PROOF: It is known that the range $m(\Sigma)$, of m, is relatively $\sigma(X, X')$-compact, [4, IV Theorem 6.1]. Consider $m_\sigma : \Sigma \to X_\sigma$. An examination of the proof of [6, Proposition 3 (i)] shows that it does not require the lcHs X there to be sequentially complete (a standing hypothesis in [6]) and hence, by this result applied to m_σ in X_σ it follows that $\chi(\mathcal{A}_\sigma) = \overline{\mathcal{A}_s(\sigma)}$. Then Lemma 2 implies the result. □

Remark 2. (i) Proposition 3 applies to a large class of spaces X, different from the spaces X admitted in Proposition 3(i) of [6] where typically the bounded sets of X are required to be metrizable for the *given* topology in X. For example, if X is a quasicomplete Suslin lcHs, then it is also Suslin for the weak topology, [8], and hence, compact subsets of X_σ are metrizable for the weak topology, [1, Chapter 9, Appendix 1, Corollary 2 to Proposition 3]. The class of lcH Suslin spaces is quite extensive, [7]; [8]. Or, if X' is weak-star separable, then compact subsets of X_σ are metrizable for $\sigma(X, X')$, [3, Proposition 3.2]. Or, if $X = Y'$ is a dual space, then certain properties of Y may imply that particular balanced, convex, $\sigma(X, Y)$-closed and bounded (or equicontinuous) subsets of X, including the balanced, closed, convex hull of $m(\Sigma)$, are $\sigma(X, Y)$-metrizable, [6, Proposition 4].

(ii) For a *particular* measure $m : \Sigma \to X$ the conclusion of Proposition 3 holds under the assumption that just $m(\Sigma)$ itself is $\sigma(X, X')$-metrizable; no particular properties of the space X are then required. □

Remark 2, Proposition 3 and [6, Proposition 3 (i)] show that there is an extensive class of spaces X with the property that $\chi(\mathcal{A}_\sigma) = \overline{\mathcal{A}}_s$, whenever $m : \Sigma \to X$ is a vector measure and $\mathcal{A} \subseteq \Sigma$ is an algebra of sets. For all further examples of vector measures m in spaces X which are known to the author (some such examples are given in [6] where X does not have any properties of the type above) the equality $\chi(\mathcal{A}_\sigma) = \overline{\mathcal{A}}_s$ also holds. This suggests the conjecture that perhaps $\chi(\mathcal{A}_\sigma) = \overline{\mathcal{A}}_s$ always holds in general. If so, then this would be an interesting result because it would follow that $\chi(\mathcal{A}_\sigma) = \overline{\mathcal{A}_s(\rho)}$, for every consistent lcH-topology ρ in X. That is, the sequential closure of $\chi(\mathcal{A})$ in $\Sigma(m)$ would be, as a subset of $\Sigma(m)$, independent of which topology $\rho(m_\rho)$ is used in $\Sigma(m)$!

In conclusion, we recall that a vector measure $m : \Sigma \to X$ is called *closed*, [4, Chapter IV], if $(\Sigma(m), \tau(m))$ is a complete topological space. It is easy to exhibit examples of vector measures m which are not closed, [4, p. 77]. However, all examples of vector measures m known to the author have the property that $\Sigma(m)$ is $\tau(m)$-*sequentially* complete; call such a vector measure σ-*closed.* It would be interesting to know whether all vector measures are necessarily σ-closed.

References

[1] Bourbaki N., *Topologie générale. II* (Nouvelle Édition), Chapitres 5 à 10, Herman, Paris, 1974.

[2] Dunford N., Schwartz J.T., *Linear operators III; spectral operators*, Wiley-Interscience, New York, 1972.

[3] Floret K., *Weakly compact sets*, Lecture Notes in Math., Vol. 801, Springer-Verlag, Berlin and New York, 1980.

[4] Kluvánek I., Knowles G., *Vector measures and control systems*, North Holland, Amsterdam, 1976.

[5] Ricker W.J., *Criteria for closedness of vector measures*, Proc. Amer. Math. Soc. **91** (1984), 75–80.

[6] Ricker W.J., *Separability of the L^1-space of a vector measure*, Glasgow Math. J. **34** (1992), 1–9.

[7] Schwartz L., *Radon measures on arbitrary topological spaces and cylindrical measures*, Oxford University Press, Bombay, 1973.

[8] Thomas G.E.F., *Integration of functions in locally convex Suslin spaces*, Trans. Amer. Math. Soc. **212** (1975), 61–81.

School of Mathematics, University of New South Wales, Sydney, NSW 2052, Australia

(*Received* January 10, 1995)