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# A remark on accessible and axiomatizable categories

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*Abstract.* For categories with equalizers the concepts "accessible" and "axiomatizable" are equivalent. This results is proved under (in fact, is equivalent to) the large-cardinal Vopěnka's principle.

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Recall that a category $\mathcal{K}$ is $\lambda$-*accessible* (see [L], [MP]), where $\lambda$ is a regular cardinal, provided that it has $\lambda$-directed colimits and satisfies the following condition

(∗) $\mathcal{K}$ has a set of $\lambda$-presentable objects whose closure under $\lambda$-directed colimits is all of $\mathcal{K}$.

Every accessible category is *axiomatizable*, i.e. equivalent to the category of all models of some (many-sorted, infinitary, first-order) theory and all homomorphisms, see [MP]. The converse is not true, in general: the axiom $(\exists\, x, y)(x \neq y)$ axiomatizes the class of all sets of at least two elements, and this category is not accessible because it does not have $\lambda$-directed colimits for any $\lambda$. However, in the present paper we prove the converse for categories with equalizers: every axiomatizable category with equalizers is accessible. This result requires the following

*Vopěnka's principle*: the category of graphs does not have a full, large, discrete subcategory.

As shown in [J], this is a large-cardinal principle: it implies the existence of measurable cardinals, and the existence of huge cardinals implies that Vopěnka's principle is consistent.

Vopěnka's principle simplifies the theory of accessible categories considerably, because assuming it we can conclude that

(i) a category is accessible iff it is *bounded*, i.e. has a small dense subcategory, and it has $\lambda$-directed colimits for some $\lambda$,

(ii) a full subcategory of an accessible category $\mathcal{K}$ is accessible iff it is closed in $\mathcal{K}$ under $\lambda$-directed colimits for some $\lambda$,

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(iii) a category is locally presentable (i.e. cocomplete and satisfying $(*)$) iff it is bounded and complete,

see [RTA]. In the present note we generalize (iii) by proving that a category is locally multipresentable iff it is bounded and has connected limits. Recall from [D] that a category is called *locally $\lambda$-multipresentable* provided that it satisfies $(*)$ and has multicolimits or, equivalently, connected limits (i.e. limits whose schemes are indecomposable).

Our main result is closely related to the unpublished theorem of E.R. Fisher, [F], that assuming Vopěnka's principle, every category of structures which has equalizers is axiomatizable.

**Theorem.** *Assuming Vopěnka's principle, the following conditions are equivalent for each category $\mathcal{K}$ with equalizers:*

(i) *$\mathcal{K}$ is axiomatizable,*
(ii) *$\mathcal{K}$ is accessible,*
(iii) *$\mathcal{K}$ is bounded.*

Proof: For each category, (ii) $\Rightarrow$ (i) is proved in [MP], and (i) $\Rightarrow$ (iii) follows from the downward Löwenheim-Skolem theorem (see [AR, 5.41]).

Assuming Vopěnka's principle, we prove (iii) $\Rightarrow$ (ii) whenever $\mathcal{K}$ has equalizers. Let $\mathcal{A}$ be a small, dense, full subcategory of $\mathcal{K}$, then the Yoneda embedding

$$E : \mathcal{K} \to \mathbf{Set}^{\mathcal{A}^{op}}$$

given by $E(K) = \hom(-, K)/\mathcal{A}^{op}$ is full, faithful, and preserves equalizers. Besides, there clearly exists a (finitary, many-sorted) signature $\Sigma$ such that $\mathbf{Set}^{\mathcal{A}^{op}}$ is equivalent to a full, reflective subcategory of the category $\mathbf{Str}\Sigma$ of $\Sigma$-structures and homomorphisms. Thus, it is sufficient to prove that every full subcategory $\mathscr{L}$ of $\mathbf{Str}\Sigma$ closed under equalizers is accessible.

By Theorem 6.17 in [AR], it is sufficient to prove that there exists a regular cardinal $\lambda$ such that $\mathscr{L}$ is closed under $\lambda$-pure subobjects for some $\lambda$. These are defined as monomorphisms $m\colon A \to B$ such that given $\lambda$-presentable objects $A'$, $B'$, and a morphism $m'\colon A' \to B'$, then for each morphism $f\colon A' \to A$ we have: if $mf$ factorizes through $m'$, then $f$ factorizes through $m'$. Moreover, for each $\lambda$-pure subobject $m\colon A \to B$ there exists a $\lambda$-directed diagram $D$ (indexed by a $\lambda$-directed poset $I$) with a colimit $(B_i \xrightarrow{b_i} B)_{i\in I}$ and there exist compatible split monomorphisms $m_i\colon A \to B_i$ $(i \in I)$ such that $m = b_i m_i$ for each $i \in I$, see Proposition 2.30 of [AR].

In order to prove the closedness of $\mathscr{L}$ under pure subobjects, we work with the full subcategory $\bar{\mathscr{L}}$ of all subobjects of objects in $\mathscr{L}$. Since $\bar{\mathscr{L}}$ is closed under subobjects, Vopěnka's principle implies that $\bar{\mathscr{L}}$ is axiomatizable in the $\lambda$-ary first-order logic $L_\lambda$ for some cardinal $\lambda$, see Proposition 6.20 of [AR]. Furthermore, Vopěnka's principle implies the existence of arbitrarily large compact cardinals, see [J], thus we can assume that the cardinal $\lambda$ under consideration is compact.

Given a $\lambda$-pure subobject $m\colon A \to B$ with $B \in \mathscr{L}$, we are going to prove that $A \in \mathscr{L}$. Let $m = b_i m_i$ $(i \in I)$ be morphisms as above, and for each $i$ let $\bar{m}_i\colon B_i \to A$ be a morphism with $\bar{m}_i m_i = id_A$. Since $I$ is a $\lambda$-directed poset, the sets $\uparrow i = \{j \in I \,|\, i \leq j\}$, $i \in I$, form a basis of a $\lambda$-complete filter and, because $\lambda$ is compact, there exists a $\lambda$-complete ultrafilter $\mathcal{U}$ containing each $\uparrow i$, $i \in I$. The class $\bar{\mathscr{L}}$, being axiomatizable in $L_\lambda$, is closed under ultraproducts indexed by $\lambda$-complete ultrafilters — thus, the ultrapower $B^{\mathcal{U}}$ lies in $\bar{\mathscr{L}}$. We can describe $B^{\mathcal{U}}$ as a $\lambda$-directed colimit of the diagram $D^*$ of all powers $B^U$ $(U \in \mathcal{U})$ and all the canonical projections $d^*_{U,V} : B^U \to B^V$ for $V \subseteq U$. Let $b^*_i : B^{\uparrow i} \to B^{\mathcal{U}}$ be the colimit maps for $i \in I$. Let

$$p_i, q_i : B_i \to B^{\uparrow i} \qquad (i \in I)$$

be the morphisms whose $j$-th components are $b_{i,j} : B_i \to B_j$ (the connecting maps of the above diagram $D$) and $m_j \bar{m}_j b_{i,j}\colon B_i \to B_j$, respectively. By colimiting, we get morphisms

$$p, q : B \to B^{\mathcal{U}},$$

and we are going to prove that the equalizer of $p, q$ is $m\colon A \to B$. Since $B$ lies in $\mathscr{L}$ and $B^{\mathcal{U}}$ is a subobject of an object of $\mathscr{L}$ (because $B^{\mathcal{U}} \in \bar{\mathscr{L}}$), it then follows that $m$ is also an equalizer of two morphisms between objects of $\mathscr{L}$. Since $\mathscr{L}$ is closed under equalizers, we conclude $A \in \mathscr{L}$, as desired.

To prove that $m$ is an equalizer of $p$ and $q$, first observe that the equality

$$b_{i,j} m_i = m_j = m_j \bar{m}_j m_j = m_j \bar{m}_j b_{i,j} m_i$$

yields, by colimiting, $pm = qm$. Next, given a morphism $h : H \to B$ with $ph = qh$, we will prove that $h$ factorizes through $m$ (necessarily uniquely). Since $\mathbf{Str}\Sigma$ is locally $\lambda$-presentable, $H$ is a $\lambda$-directed colimit of $\lambda$-presentable objects. Thus, it suffices to prove that $p$ factorizes through $m$ under assumption that $H$ is a $\lambda$-presentable object. Since $B$ is a $\lambda$-directed colimit of $D$, there exists $i \in I$ and $h' : H \to B_i$ with $h = b_i h'$. From

$$b^*_i(p_i h') = p b_i h' = q b_i h' = b^*_i(q_i h')$$

we conclude, since $B^{\mathcal{U}}$ is a $\lambda$-directed colimit of $D^*$ and $H$ is $\lambda$-presentable, that there exists $U \in \mathcal{U}$, $U \subseteq \uparrow i$ with

$$d^*_{\uparrow i, U}(p_i h') = d^*_{\uparrow i, U}(q_i h')\,.$$

Choosing any $j \in U$ and composing the last equality with the $j$-th projection of $B^U$ yields

$$b_{i,j} h' = m_j \bar{m}_j b_{i,j} h'\,.$$

Consequently, $h$ factorizes through $m$ (as required):

$$h = b_i h' = b_j b_{i,j} h' = b_j m_j \bar{m}_j b_{i,j} h' = m \bar{m}_j b_{i,j} h'\,.$$

$\square$

**Corollary.** *Assuming Vopěnka's principle, a category is locally multipresentable iff it is bounded and has connected limits.*

In fact, each bounded category with connected limits is accessible, thus, it is locally multipresentable.

**Remark.** Both the theorem and the corollary are actually equivalent to Vopěnka's principle. In fact, assuming the negation of Vopěnka's principle, a complete bounded category which is not axiomatizable is presented in [AR, Remark 6.20].

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