Radical ideals and coherent frames

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Abstract. It follows from Stone Duality that Hochster's results on the relation between spectral spaces and prime spectra of rings translate into analogous, formally stronger results concerning coherent frames and frames of radical ideals of rings. Here, we show that the latter can actually be obtained without Stone Duality, proving them in Zermelo-Fraenkel set theory and thereby sharpening the original results of Hochster.

Keywords: coherent frame or locale, radical ideal, prime spectrum, spectral space, support on a ring, Boolean powers

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A celebrated result of Hochster [1] says that every spectral space X is homeomorphic to the prime spectrum $\operatorname{Spec} A$ of a commutative ring A with unit, and that the correspondence $X \rightsquigarrow A$ such that $X \cong \operatorname{Spec} A$ can even be made functorial on certain categories of spectral spaces. From this one obtains, using the category equivalence between coherent frames and bounded distributive lattices together with Stone Duality for the latter, that every coherent frame L is isomorphic to the frame $RIdA$ of radical ideals of a commutative ring A with unit, again with a certain measure of functoriality for the correspondence $L \rightsquigarrow A$.

Now, given that frames — or alternatively their formal duals, locales — have long been recognized as more fundamental than spaces, it becomes a natural challenge to derive the latter result without recourse to Stone Duality. The purpose of this paper is to do just that. Since Stone Duality is equivalent to the Prime Ideal Theorem for Boolean algebras, we shall achieve this by arguing entirely within Zermelo-Fraenkel set theory.

The paper is organized as follows: After Section 0 on background, we discuss the two basic tools employed here, supports (Section 1) and Boolean powers (Section 2). Supports were originally introduced by Joyal to provide a pointfree version of the prime spectrum, and hence it is not surprising that they should become useful here. Boolean powers, on the other hand, play the rôle of rings of functions when there is no set that may serve as their domain.

After these preparations, obtaining a ring A with prescribed $RIdA$ proceeds in two different stages: we first deal with the case that the given coherent frame is finite (Section 3), and then apply the resulting construction to the finite coherent subframes of an arbitrary coherent frame to prove the result in general (Section 4). In addition, we show that the process we use is functorial on the category of coherent frames and coherent embeddings, and that every coherent frame homomorphism occurs in the image of the functor RId.

It should be noted that the results presented here could also be derived from those of Hochster [1] by using appropriate properties of the functor RId together with *finite* Stone Duality (which *does* hold in Zermelo-Fraenkel set theory), as will be discussed in more detail in the Concluding Remarks. Nonetheless, there seems to be some merit in giving a proof *ab initio* which places the argument entirely into the algebraic context.

0. Background

For general facts concerning frames and various related ideas we refer to Johnstone [3]. Here we recount some special results and fix some notation and terminology.

If L is any frame then KL will be the join-subsemilattice of its *compact* elements, that is, the $c \in L$ such that, for any $S \subseteq L$, $c \leq \bigvee S$ implies $c \leq \bigvee T$ for some finite $T \subseteq S$. Then, L is called *coherent* whenever KL generates L and is a sublattice, including the unit e of L. Further, for coherent frames L and M, a homomorphism $h : L \to M$ is called *coherent* provided it maps KL into KM, and **CohFrm** will be the resulting category.

The correspondence $L \rightsquigarrow KL$ then obviously determines a functor: K : **CohFrm** $\to$ **D**, where the latter is the category of bounded distributive lattices; moreover, K is a category equivalence, with inverse given by the ideal lattice functor $\mathcal{I}$.

Every bounded distributive lattice A is a sublattice of a Boolean algebra $B(A) \supseteq A$, the *Boolean envelope* of A, such that the identical embedding $A \to B(A)$ is the reflection map to Boolean algebras, that is, any homomorphism from A into a Boolean algebra lifts uniquely to $B(A)$. It should be emphasized that $B(A)$ can be constructed within the congruence lattice of A, and as a result both, its existence and its reflection property, are certainly assured in Zermelo-Fraenkel set theory.

Regarding the category **Frm** of frames in general and the category **Top** of topological spaces, we have the spectrum functor Σ : **Frm** $\to$ **Top** and the functor $\mathcal{O}$: **Top** $\to$ **Frm**, contravariant and adjoint to each other on the right. Here, ΣL is the space of all homomorphisms $\xi : L \to 2$, with open sets $\Sigma_a = \{\xi \in \Sigma L \mid \xi(a) = 1\}$, and $\mathcal{O}X$ the frame of open subsets of X, with the obvious effects on maps.

A space X is called *spectral* whenever the frame $\mathcal{O}X$ is coherent and X is sober, that is, the adjunction map $X \to \Sigma\mathcal{O}X$ is a homeomorphism. For spectral spaces X and Y, a *spectral map* from X to Y is a continuous map $f : X \to Y$ such that $\mathcal{O}f : \mathcal{O}Y \to \mathcal{O}X$ is coherent, meaning that $f^{-1}(U)$ is compact whenever $U \subseteq Y$ is compact open. Letting **Spec** $\subseteq$ **Top** be the corresponding category, $\mathcal{O}$ induces a functor **Spec** $\to$ **CohFrm** by definition, and assuming the Boolean Prime Ideal Theorem (PIT) one proves this is a dual equivalence, with inverse provided by Σ.

In that situation, the composite (dual) equivalences

$$\mathbf{Spec} \underset{\Sigma}{\overset{\mathcal{O}}{\leftrightarrows}} \mathbf{CohFrm} \underset{\mathcal{I}}{\overset{K}{\leftrightarrows}} \mathbf{D}$$

amount to Stone Duality, and this in turn trivially implies PIT.

Now, concerning rings, we let **Ann** be the category of commutative rings with unit, using the term "ring" always in this sense. For any ring A, a *radical ideal* in A is an ideal J such that for any $x \in A$ and natural n, $x^n \in J$ implies $x \in J$. In particular, for any ideal I in A,

$$r(I) = \{x \in A \mid \text{ some } x^n \in I\}$$

is the smallest radical ideal containing I. We put

$$[a_1, \ldots, a_n] = r(Aa_1 + \cdots + Aa_n)$$

for any $a_1, \ldots, a_n \in A$. Further $RIdA$ will be the set of all radical ideals of A, partially ordered by inclusion, evidently closed under arbitrary intersections and hence a complete lattice; more specifically, $RIdA$ is a frame, as one easily derives from certain properties of the above operator r. In addition, one has the following:

the compact elements of $RIdA$ are exactly the finitely generated radical ideals $[a_1, \ldots, a_n]$,
for any $a, b \in A$, $[a] \wedge [b] = [ab]$ since $x^n \in Aa$ and $x^m \in Ab$ implies $x^{n+m} \in Aab$,
and as a consequence
$RIdA$ is coherent.

Further, the correspondence $A \rightsquigarrow RIdA$ is functorial, the effect on a ring homomorphism $h : A \to B$ being the map taking each $J \in RIdA$ to the radical ideal of B generated by $h[J]$.

Another, perhaps more familiar, (contravariant) functor on **Ann** is the prime spectrum functor Spec : **Ann** $\to$ **Top** for which $\operatorname{Spec} A$ is the space of prime ideals $P \subsetneq A$, with basic open sets

$$W_a = \{P \in \operatorname{Spec} A \mid a \notin P\} \qquad (a \in A)$$

and $\operatorname{Spec} h : \operatorname{Spec} B \to \operatorname{Spec} A$ takes inverse images of prime ideals, for any ring homomorphism $h : A \to B$. Actually, Spec may be defined in terms of RId: one shows that the prime ideals of A are exactly the prime elements of the frame $RIdA$, and this readily leads to a natural equivalence $\operatorname{Spec} \cong \Sigma RId$.

Now, if one assumes PIT then one can prove that $\operatorname{Spec} A$ is always a spectral space and $\operatorname{Spec} h$ always a spectral map so that one actually has a functor Spec : **Ann** $\to$ **Spec**; moreover, in this situation $\mathcal{O}\Sigma \cong Id$ on **CohFrm**, as remarked earlier, and consequently $RId \cong \mathcal{O}\operatorname{Spec}$. The significance of RId therefore lies in the fact that it represents the frame of open sets of the prime spectrum without any reference to the latter.

1. Supports

Here, we consider a general device which, though not originally introduced for that purpose, can be used to detect isomorphisms $RIdA \to L$ for $A \in$ **Ann** and $L \in$ **CohFrm**.

We begin with an auxiliary result on coherent frames.

Lemma 1. *A coherent frame homomorphism $h : L \to M$ is one-one whenever $h\,|\,KL$ is one-one.*

Proof: Given $h(a) \leq h(b)$, consider any compact $c \leq a$. Then $h(c) \leq h(b)$, and by coherence there exist compact $d \leq b$ such that $h(c) \leq h(d)$, implying $c \leq d$ by hypothesis and therefore $c \leq b$. Hence, again by coherence, $a \leq b$, and it then follows that $h(a) = h(b)$ implies $a = b$, as claimed. $\square$

Below, A is any ring and L any bounded lattice.

Definition 1. A *support* on A, with values in L, is a map $d : A \to L$ such that

(S1) $\quad d(ab) = d(a) \wedge d(b), \qquad d(1) = e.$

(S2) $\quad d(a+b) \leq d(a) \vee d(b), \qquad d(0) = 0.$

An obvious example of a support is given by $A = C(X)$, the ring of continuous real-valued functions on a space X, with

$$d(u) = \mathrm{Coz}(u) = \{x \in X \,|\, u(x) \neq 0\}$$

in the lattice $\mathcal{O}X$ of open subsets of X. Somewhat analogously, one has, for any A, the support $d : A \to \mathcal{O}(\mathrm{Spec}\, A)$ such that $d(a) = W_a$. Finally, and most important for our purpose, the map $A \to RIdA$ taking each $a \in A$ to its principal radical ideal $[a]$ is a support, as witnessed by some earlier calculations. We shall call this the *radical support* on A.

The following familiar result describes the crucial property of the radical support; we include a proof for completeness' sake.

Lemma 2. *Any support $d : A \to L$ with values in a frame determines a unique frame homomorphism $\tilde{d} : RIdA \to L$ such that $\tilde{d}([a]) = d(a)$.*

Proof: Put $\tilde{d}(J) = \bigvee\{d(a) \,|\, a \in J\}$. Then obviously $\tilde{d}(0) = 0$, $\tilde{d}(A) = e$, and for any radical ideals I and J of A,

$$\tilde{d}(I) \wedge \tilde{d}(J) = \bigvee\{d(a) \wedge d(b) \,|\, a \in I, b \in J\} = \bigvee\{d(ab) \,|\, a \in I, b \in J\} \leq h(I \cap J),$$

and hence equality, the reverse inequality being trivial. Next, for any updirected $\mathcal{X} \subseteq RIdA$,

$$\begin{aligned}\tilde{d}(\bigvee \mathcal{X}) = \tilde{d}(\bigcup \mathcal{X}) = \bigvee\{d(a) \,|\, a \in \bigcup \mathcal{X}\} = \bigvee\{\bigvee\{d(a) \,|\, a \in J\} \,|\, J \in \mathcal{X}\} \\ = \bigvee \tilde{d}[\mathcal{X}].\end{aligned}$$

Finally, for the case of binary join, note first that $a \in I \vee J$ iff $a^n \in I + J$ for some natural n, but also $d(a^n) = d(a)$; hence

$$\tilde{d}(I \vee J) = \bigvee \{d(x+y) \mid x \in I, y \in J\} \leq \bigvee \{d(x) \vee d(y) \mid x \in I, y \in J\}$$

showing that $\tilde{d}(I \vee J) \leq \tilde{d}(I) \vee \tilde{d}(J)$, the non-trivial part of the desired identity. Hence $\tilde{d}$ is a frame homomorphism, and obviously $\tilde{d}([a]) = d(a)$, which implies uniqueness since the $[a]$ generate $RIdA$. □

This lemma says that the radical support on a ring A is *universal* among all frame-valued supports on A. It also says that the bounded distributive lattice of all *finitely generated* radical ideals is exactly the bounded distributive lattice generated by elements $d(a)$, $a \in A$, subject to the identities (S1) and (S2), where the latter is sometimes taken as a definition of the universal support (Johnstone [3, V. 3]). Further, we note in passing that the universality of the radical support provides a particularly suggestive proof of the functoriality of $RIdA$: for any ring homomorphism $f : A \to B$, we have a commuting square

$$\begin{tikzcd} A \arrow[r, "{[\cdot]_A}"] \arrow[d, "f"'] & RIdA \arrow[d, "{([\cdot]_B f)^\sim}"] \\ B \arrow[r, "{[\cdot]_B}"'] & RIdB \end{tikzcd}$$

since $[\cdot]_B f : A \to RId$ is evidently a support.

We are interested in conditions on a support $d : A \to L$ which ensure that $\tilde{d} : RIdA \to L$ is an isomorphism.

Definition 2. A support $d : A \to L$, where L is a frame, is called

coherent if all $d(a)$, $a \in A$, are compact,
full if each $c \in KL$ is equal to some $d(a_1) \vee \cdots \vee d(a_n)$,
principal if, for any a and b in A, there exist $c \in Aa + Ab$ such that $d(c) = d(a) \vee d(b)$,
and
faithful if, for any $a, b \in A$, $d(a) \leq d(b)$ implies $[a] \subseteq [b]$. Further, d is called
perfect if it has all these properties.

Now, we have the desired result:

Lemma 3. *For any coherent frame L and perfect support $d : A \to L$, $\tilde{d} : RIdA \to L$ is an isomorphism.*

Proof: Since d is coherent and full, $\tilde{d}$ is onto. Hence it will be enough, by Lemma 1, to show that $\tilde{d}$ is one-one on compact elements, that is, on the finitely generated radical ideals. Now, $\tilde{d}([a_1, \ldots, a_n]) = \tilde{d}([b_1, \ldots, b_m])$ means that $d(a_1) \vee$

$\cdots \vee d(a_n) = d(b_1) \vee \cdots \vee d(b_m)$, and using induction and the fact that d is principal we obtain $c \in Ab_1 + \cdots + Ab_m$ such that $d(b_1) \vee \cdots \vee d(b_m) = d(c)$; then, for each i, $d(a_i) \leq d(c)$, hence $[a_i] \subseteq [c]$ since d is faithful, and therefore

$$[a_1, \ldots, a_n] \subseteq [c] \subseteq [b_1 \ldots, b_m].$$

By symmetry, this implies the desired identity. $\square$

Next, we consider the effect of the functor RId on homomorphisms connecting given supports.

Lemma 4. *For any commuting square*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "d_A"'] & B \arrow[d, "d_B"] \\ L \arrow[r, "h"'] & M \end{tikzcd}$$

where f is a ring homomorphism, h a frame homomorphism, and d_A and d_B are supports, the corresponding square of frame homomorphisms

$$\begin{tikzcd} RIdA \arrow[r, "RIdf"] \arrow[d, "\tilde{d}_A"'] & RIdB \arrow[d, "\tilde{d}_B"] \\ L \arrow[r, "h"'] & M \end{tikzcd}$$

commutes.

Proof: In the augmented diagram

$$\begin{tikzcd} A \arrow[rrr, "f"] \arrow[dr, "{[\cdot]_A}"] \arrow[ddd, "d_A"'] & & & B \arrow[dl, "{[\cdot]_B}"'] \arrow[ddd, "d_B"] \\ & RIdA \arrow[r, "RIdf"] \arrow[ddl, "\tilde{d}_A"] & RIdB \arrow[ddr, "\tilde{d}_B"'] & \\ & & & \\ L \arrow[rrr, "h"'] & & & M \end{tikzcd}$$

the outer square, the upper square, and the two triangles commute so that

$$(h\tilde{d}_A)[\cdot]_A = (\tilde{d}_B RIdf)[\cdot]_A,$$

and since the image of $[\cdot]_A$ generates $RIdA$, this proves the claim. $\square$

We close with a further result concerning the functor RId, also contained in Vermeulen [4], proved here for the sake of completeness.

Lemma 5. *RId preserves updirected colimits.*

Proof: Let I be any updirected partially ordered set, and

$$(*) \qquad f_{ik} : A_i \to A_k \quad (i \leq k \text{ in } I)$$

an I-indexed diagram in **Ann** with colimit maps $f_i : A_i \to A$. Then, in order to see that the corresponding $RIdf_i : RIdA_i \to RIdA$ are the colimit maps in **CohFrm** of the RId-image of the diagram $(*)$, it will be sufficient to show that any family $d_i : A_i \to L$ of supports compatible with the maps in $(*)$ determines a support $d : A \to L$ such that $df_i = d_i$ for all i: given any coherent homomorphisms $h_i : RIdA_i \to L$ compatible with the homomorphisms

$$RIdf_{ik} : RIdA_i \to RIdA_k,$$

apply this to $d_i = h_i[\cdot]_{A_i}$ and use Lemma 4. Now, by a familiar property of updirected colimits of finitary algebras,

$$\operatorname{Ker}(f_i) = \bigcup\{\operatorname{Ker}(f_{ik}) \mid k \geq i\};$$

hence d_i is constant on the corresponding cosets so that it determines a support $\overline{d}_i : \operatorname{Im}(f_i) \to L$. Here, $\overline{d}_i = \overline{d}_k \mid \operatorname{Im}(f_i)$ for all $k \geq i$, and since $A = \bigcup\{\operatorname{Im}(f_i) \mid i \in I\}$ the union of the $\overline{d}_i$ is a support $d : A \to L$, obviously of the desired kind. $\square$

2. Boolean powers

Here, we introduce certain rings, obtained as a particular case of a general construction in Universal Algebra.

Given a Boolean algebra B and a field F, the *Boolean power of F by B* is the ring Φ with *elements* $\alpha : F \to B$ such that

(1) $\alpha(x) \wedge \beta(y) = 0$ for all $x \neq y$ in F.
(2) $\alpha(x) = 0$ for all but finitely many $x \in F$.
(3) $\bigvee\{\alpha(x) \mid x \in F\} = e$.

and *operations* $+$, $-$, $\cdot$, zero $\mathbf{0}$, and unit $\mathbf{1}$ given by

$$(\alpha \stackrel{\pm}{\cdot} \beta)(x) = \bigvee\{\alpha(y) \wedge \beta(z) \mid x = y \stackrel{\pm}{\cdot} z\}$$
$$\mathbf{0}(0) = e, \quad \mathbf{1}(1) = e.$$

It is a familiar fact that the Boolean powers of any finitary algebra A in the sense of Universal Algebra satisfy all identities that hold in A, and hence Φ is indeed a commutative ring with unit.

Note that, for the Boolean algebra $\mathcal{B}X$ of the open-closed subsets of a Boolean (= zero-dimensional compact Hausdorff) space X, the Boolean power of F by $\mathcal{B}X$ is isomorphic to the ring $C(X, F)$ of all continuous F-valued functions on X, F taken discrete, with the usual pointwise operations, the isomorphism taking

$u \in C(X,F)$ to $\alpha \in \Phi$ such that $\alpha(x) = u^{-1}\{x\}$ for each $x \in F$. Of course, using the Stone Representation Theorem, *every* Boolean algebra is of the form $\mathcal{B}X$, and then any Φ is represented as a $C(X,F)$. The crucial point here is that even without this representation we can establish all the properties for these rings which are needed for our purpose.

Rings of functions on a set with values in a field have a natural support, given by the corresponding cozero sets. We shall now show that there is a perfect analogue to this in the case of the Boolean powers Φ.

Define $d : \Phi \to B$ by

$$d(\alpha) = \bigvee\{\alpha(x) \mid x \neq 0\}$$

for each $\alpha \in \Phi$. Also, recall that a field is called *real* whenever $x_1^2 + \cdots x_n^2 = 0$ implies $x_i = 0$ for all i. Then we have

Lemma 6. *d is a support, and if F is a real field, then $d(\alpha^2+\beta^2) = d(\alpha) \vee d(\beta)$ for all $\alpha, \beta \in \Phi$.*

PROOF: For any $\alpha, \beta \in \Phi$,

$$d(\alpha\beta) = \bigvee\{\alpha(y) \wedge \beta(z) \mid yz \neq 0\} = \bigvee\{\alpha(y) \wedge \beta(z) \mid x, y \neq 0\} = d(\alpha) \wedge d(\beta),$$

and $d(\mathbf{1}) = e$ as $\mathbf{1}(1) = e$. Further,

$$d(\alpha+\beta) = \bigvee\{\alpha(y) \wedge \beta(z) \mid y+z \neq 0\} \leq d(\alpha) \wedge d(\beta)$$

since $y + z \neq 0$ implies $y \neq 0$ or $z \neq 0$ and then $\alpha(y) \leq d(\alpha)$ or $\beta(z) \leq d(\beta)$ so that $\alpha(y) \wedge \beta(z) \leq d(\alpha) \vee d(\beta)$. Finally $d(\mathbf{0}) = 0$ since $\mathbf{0}(0) = e$.

For the second part of the assertion, we have

$$\begin{aligned} d(\alpha^2+\beta^2) &= \bigvee\{\alpha(y_1) \wedge \alpha(y_2) \wedge \beta(z_1) \wedge \beta(z_2) \mid y_1y_2 + z_1z_2 \neq 0\} \\ &= \bigvee\{\alpha(y) \wedge \beta(z) \mid y^2 + z^2 \neq 0\} \\ &= \bigvee\{\alpha(y) \wedge \beta(z) \mid y \neq 0 \text{ or } z \neq 0\} \\ &= d(\alpha) \vee d(\beta), \end{aligned}$$

the second step since $y_1 \neq y_2$ or $z_1 \neq z_2$ produce zero terms, and the third step because F is real.

On any subring of Φ, the restriction of the above $d : \Phi \to B$ will be called the *standard* support. Note that, whenever the field F is real, this will always be principal in view of the identity $d(\alpha^2+\beta^2) = d(\alpha) \vee d(\beta)$. Of course, if the Boolean algebra is $\mathcal{B}X$ for some Boolean space X, then the isomorphism between Φ and $C(X,F)$ makes the standard support on Φ correspond exactly to the support $C(X,F) \to \mathcal{B}X$ given by taking cozero sets.

Next, we introduce a conditional division in Φ analogous to that in the case of functions.

Definition 3. For $\alpha, \beta \in \Phi$ such that $d(\alpha) \leq d(\beta)$, $\alpha\#\beta$ is given by

$$\alpha\#\beta(x) = \bigvee\{\alpha(y) \wedge \beta(z) \mid x = yz^{-1},\ z \neq 0\} \quad (0 \neq x \in F)$$
$$\alpha\#\beta(0) = \alpha(0).$$

We have to check that $\alpha\#\beta$ does in fact belong to Φ.

Given any $x \neq x'$, consider first the case when $x \neq 0 \neq x'$. Then

$$\begin{aligned} &\alpha\#\beta(x) \wedge \alpha\#\beta(x') \\ &= \bigvee\{\alpha(y) \wedge \beta(z) \wedge \alpha(y') \wedge \beta(z') \mid xz = y,\ x'z' = y',\ z \neq 0 \neq z'\} \\ &= 0 \end{aligned}$$

since $y \neq y'$ or $z \neq z'$ because $x \neq x'$. On the other hand, for $x \neq 0$ and $x' = 0$ we have

$$\begin{aligned} \alpha\#\beta(x) \wedge \alpha\#\beta(0) &= \bigvee\{\alpha(y) \wedge \beta(z) \wedge \alpha(0) \mid x = yz^{-1},\ z \neq 0\} \\ &= 0 \end{aligned}$$

since $x \neq 0$ implies $y \neq 0$. In all, this establishes the first condition.

Next, $\alpha\#\beta(x) \neq 0$ means that $\alpha(y) \wedge \beta(z) \neq 0$ for some y and $z \neq 0$ such that $x = yz^{-1}$, and since there are only finitely many y and z at all for which $\alpha(y), \beta(z) \neq 0$, there can only be finitely many such x.

Finally,

$$\begin{aligned} \bigvee\{\alpha\#\beta(x) \mid x \in F\} &= \alpha(0) \vee \bigvee\{\alpha(y) \wedge \beta(z) \mid y, z \neq 0\} \\ &= \alpha(0) \vee (d(\alpha) \wedge d(\beta)) = \alpha(0) \vee d(\alpha) = e, \end{aligned}$$

the step next to the last because $d(\alpha) \leq d(\beta)$. □

The following gives some of the basic properties of this operation, showing in particular that it is indeed a form of division.

Lemma 7. (1) *$d(\alpha\#\beta) = d(\alpha)$.*
(2) *$(\alpha\#\beta)\beta = \alpha$ whenever $d(\alpha) \leq d(\beta)$.*
(3) *$(\beta\gamma)\#\beta = \gamma$ whenever $d(\gamma) \leq d(\beta)$.*

Proof: (1) By definition, $\alpha\#\beta(0) = \alpha(0)$, and for any $\gamma \in \Phi$, $d(\gamma)$ is obviously the complement of $\gamma(0)$.

(2) For any $x \neq 0$ in F,

$$\begin{aligned} (\alpha\#\beta)\beta(x) &= \bigvee\{\alpha\#\beta(y) \wedge \beta(z) \mid x = yz\} \\ &= \bigvee\{\alpha(u) \wedge \beta(v) \wedge \beta(z) \mid x = uv^{-1}z,\ v \neq 0\} \\ &= \bigvee\{\alpha(x) \wedge \beta(z) \mid z \neq 0\} \\ &= \alpha(x) \wedge d(\beta) = \alpha(x), \end{aligned}$$

the third step since $\beta(v) \wedge \beta(z) = 0$ whenever $v \neq z$, and the final step because $\alpha(x) \leq d(\alpha) \leq d(\beta)$. Further,

$$\begin{aligned}(\alpha\#\beta)\beta(0) &= \bigvee\{\alpha(0) \wedge \beta(z) \,|\, z \in F\} \vee \bigvee\{\alpha\#\beta(y) \wedge \beta(0) \,|\, y \in F\} \\ &= \alpha(0) \vee \beta(0) = \alpha(0),\end{aligned}$$

the last step since $d(\alpha) \leq d(\beta)$ implies $\beta(0) \leq \alpha(0)$.

(3) For any $x \neq 0$ in F

$$\begin{aligned}(\beta\gamma)\#\beta(x) &= \bigvee\{\beta\gamma(y) \wedge \beta(z) \,|\, x = yz^{-1},\ z \neq 0\} \\ &= \bigvee\{\beta(u) \wedge \gamma(v) \wedge \beta(z) \,|\, x = uvz^{-1},\ z \neq 0\} \\ &= \bigvee\{\gamma(x) \wedge \beta(z) \,|\, z \neq 0\} \\ &= \gamma(x) \wedge d(\beta) = \gamma(x),\end{aligned}$$

the last step since $\gamma(z) \leq d(\gamma) \leq d(\beta)$. Further

$$\begin{aligned}(\beta\gamma)\#\beta(0) = \beta\gamma(0) &= \bigvee\{\beta(y) \wedge \gamma(z) \,|\, 0 = yz\} \\ &= \beta(0) \vee \gamma(0) = \gamma(0),\end{aligned}$$

again because $d(\gamma) \leq d(\beta)$. □

We close this section with the remark that, for any homomorphism $h : C \to B$ between Boolean algebras, one has a homomorphism $\Phi \to \Psi$ from the Boolean power Φ of F by C to the Boolean power Ψ of F by B, taking any $\alpha \in \Phi$ to the composite $h\alpha$: clearly, $h\alpha \in \Psi$ and the correspondence $\alpha \rightsquigarrow h\alpha$ obviously preserves all the operations.

3. Finite distributive lattices

In this section, we construct, for any given finite distributive lattice M, a ring whose lattice of radical ideals is isomorphic to M. We begin with some lattice theoretical preparations.

Let H be the set of all lattice homomorphisms $\xi : M \to 2$, with pointwise partial order, and $B \supseteq M$ the Boolean envelope of M. Then the properties of the latter ensure that each $\xi \in H$ uniquely extends to a homomorphism $B \to 2$, again denoted by ξ. The following provides a characterization of M within B in terms of the action of the $\xi \in H$.

Lemma 8. *For any $c \in B$, $c \in M$ iff $\xi(c) \leq \zeta(c)$ for all $\xi \leq \zeta$ in H.*

Proof: For each $a \in M$, let $H_a = \{\xi \in H \,|\, \xi(a) = 1\}$. Then, for any $\xi \in H$,

$$\uparrow \xi = \{\zeta \in H \,|\, \xi \leq \zeta\} = H_s,$$

where $s = \bigwedge\{t \in M \mid \xi(t) = 1\} : \xi(s) = 1$ since M is finite, hence $\uparrow \xi \subseteq H_s$ while $\zeta(s) = 1$ implies that $\zeta(t) = 1$ whenever $\xi(t) = 1$, for all $t \in M$, but this means $\xi \leq \zeta$, showing that $H_s \subseteq \uparrow \xi$.

Next, for $c \in B$ as in the lemma, put

$$U = \{\xi \in H \mid \xi(c) = 1\}.$$

Then $\xi \in U$ implies $\zeta \in U$ for all $\zeta \geq \xi$ by hypothesis, and therefore

$$U = \bigcup\{\uparrow \xi \mid \xi \in U\}.$$

Now, the map $a \rightsquigarrow H_a$ is a lattice homomorphism from M into the power set of H, and hence $U = H_a$ for some $a \in M$, again by finiteness. This says that $\xi(c) = 1$ iff $\xi(a) = 1$, for all $\xi \in H$, and since the $\xi \in H$ separate the elements of B we conclude that $c = a$ and thus $c \in M$.

Now, let $F = \boldsymbol{Q}(z_a \mid a \in M)$ be the field of rational functions over $\boldsymbol{Q}$ in indeterminates z_a such that $a \rightsquigarrow z_a$ is one-one, and Φ the Boolean power of F by B. Note that F is a real field so that the standard support is principal on each subring of Φ.

Any $\alpha \in \Phi$ gives rise to an F-valued function $\widehat{\alpha}$ on H such that

$$\widehat{\alpha}(\xi) = x \quad \text{iff} \quad \xi(\alpha(x)) = 1,$$

for all $\xi \in H$ and $x \in F$, and the correspondence $\alpha \rightsquigarrow \widehat{\alpha}$ is an isomorphism $\Phi \to F^H$, essentially the finite case of the isomorphism $C(X, F) \to \Phi$ mentioned earlier. Where convenient, we shall also use $\alpha(\xi)$ instead of $\widehat{\alpha}(\xi)$, par abus de langage. We note some simple facts: For any $\alpha, \beta \in \Phi$ and $\xi, \zeta \in H$:

(i) $\xi(d(\alpha)) = 1$ iff $\alpha(\xi) \neq 0$.

(ii) If $d(\alpha) \in M$ and $\alpha(\xi) \neq 0$, then also $\alpha(\zeta) \neq 0$ for all $\zeta \geq \xi$.

(iii) If $d(\alpha) \leq d(\beta)$, then

$$\alpha \# \beta(\xi) = \begin{cases} \alpha(\xi)/\beta(\xi) & (\beta(\xi) \neq 0) \\ 0 & (\beta(\xi) = 0) \end{cases}$$

where the top part results from the identity $(\alpha\#\beta)\beta = \alpha$, and the bottom part from the fact that $\beta(0) \leq \alpha(0) = \alpha\#\beta(0)$ and $\beta(\xi) = 0$ means $\xi(\beta(0)) = 1$.

The desired ring A for which $RIdA \cong M$ will be constructed as a subring of Φ such that the standard support on A maps A into M and is perfect. Since M is finite and d is principal on any subring of Φ, this just means that d must map A *onto* M and be *faithful*.

An obvious subring of Φ which is mapped onto M by the standard support is given as follows: For each $a \in M$, let $\sigma_a \in \Phi$ be defined by

$$\sigma_a(z_a) = a \quad \text{and} \quad \alpha_a(0) = \sim a,$$

and put $P = \boldsymbol{Q}[\sigma_a \,|\, a \in M]$ where $\boldsymbol{Q} \subseteq \Phi$ is understood as the ring of constants mapping λ to e, for every rational number λ. Note that $\sigma_{a_1}\sigma_{a_2}\dots\sigma_{a_n} = \mathbf{0}$ iff $a_1 \wedge a_2 \wedge \dots \wedge a_n = 0$. Further, for any $\xi \in H$,

$$\widehat{\sigma}_a(\xi) = \begin{cases} z_a & (\xi(a) = 1) \\ 0 & (\xi(a) = 0), \end{cases}$$

that is, $\widehat{\sigma}_a$ is the F-valued function on H with value z_a on H_a and value 0 on its complement, a kind of characteristic function of H_a with its own distinctive label.

For any $\tau \in P$, $\tau = \Sigma\lambda\sigma_{a_1}^{k_1}\cdots\sigma_{a_n}^{k_n}$ with non-zero distinct monomials $\sigma_{a_1}^{k_1}\cdots\sigma_{a_n}^{k_n}$ and non-zero coefficients $\lambda \in \boldsymbol{Q}$. The individual $\lambda\sigma_{a_1}^{k_1}\cdots\sigma_{a_n}^{k_n}$ occurring here are uniquely determined by τ if the corresponding sequences $(a_1, k_1), \dots, (a_n, k_n)$ are taken distinct and all exponents $k_i > 0$: if $\Sigma\lambda\sigma_{a_1}^{k_1}\cdots\sigma_{a_n}^{k_n} = 0$, focus on one particular term, say $\mu\sigma_{b_1}^{\ell_1}\cdots\sigma_{b_m}^{\ell_m}$, and take any $\xi \in H$ such that $\xi(b_1 \wedge \dots \wedge b_m) = 1$; then

$$\Sigma\lambda\xi(a_1 \wedge \dots \wedge a_n)z_{a_1}^{k_1}\cdots z_{a_n}^{k_n} = 0$$

where no cancellations are possible since the monomials $z_{a_1}^{k_1}\cdots z_{a_n}^{k_n}$ are all distinct, and as $\mu z_{b_1}^{k_1}\cdots z_{b_m}^{k_n}$ occurs here, we have $\mu = 0$. We call the elements $\lambda\sigma_{a_1}^{k_1}\cdots\sigma_{a_n}^{k_n}$ the *summands* of τ and the corresponding expression for τ its *reduced representation*.

To determine the values of the standard support on P, note that trivially $d(\sigma_a) = a$ and hence $d(\sigma_{a_1}^{k_1}\cdots\sigma_{a_n}^{k_n}) = a_1 \wedge \dots \wedge a_n$. In general one has

$$(\#) \qquad d(\tau) = \bigvee\{a_1 \wedge \dots \wedge a_n \mid \text{ all summands } \lambda\sigma_{a_1}^{k_1}\cdots\sigma_{a_n}^{k_n} \text{ of } \tau\}.$$

Here, $\leq$ is obvious by the general properties of supports, and $\geq$ follows by considering $\tau(x)$ for $x = \Sigma\lambda z_{a_1}^{k_1}\cdots z_{a_n}^{k_n}$ corresponding to the reduced representation of $\tau : d(\tau) \geq \tau(x)$ since $x \neq 0$, and $\tau(x)$ is clearly above the join in (#).

The ring P is obviously the most natural starting point for constructing a ring A for which $RIdA \cong M$, but unfortunately P itself is not good enough: *Whenever $M \neq 2$ d is not faithful on P.* If $0 < a < b$ in M, then $d(\sigma_a) \leq d(\sigma_b)$ but if $\sigma_a \in [\sigma_b]$, that is, $\sigma_a^n = \tau\sigma_b$ for some natural n and $\tau \in P$, then $z_a^n = \widehat{\tau}(\xi)z_b$ for any $\xi \in H$ with $\xi(a) = 1$, and this cannot hold in the polynomial ring $\boldsymbol{Q}[z_a \,|\, a \in M]$. On the other hand, it is not possible to mend this defect just by iteratively adjoining *all* $\alpha\#\beta$ for α, β, with $d(\alpha) \leq d(\beta)$: already at the first step, the crucial property that the standard support takes only values in M is lost. For instance, if $\gamma = \sigma_a\#\sigma_a$ then $\gamma(1) = a$ and $\gamma(0) = \sim a$ so that $d(\mathbf{1} - \gamma) = \sim a$, and unless M is Boolean there are $a \in M$ for which $\sim a \notin M$. Of course, removing an instance of violated faithfulness does not require adjoining $\alpha\#\beta$, for $d(\alpha) \leq d(\beta)$ — any other $\alpha^n\#\beta$ would do equally well; on the other hand, whenever $d(\alpha) \leq d(\beta)$ and $\alpha^n = \tau\beta$ for some n and τ, then necessarily $\tau = \alpha^n\#\beta$, and hence adjoining some suitable $\alpha^n\#\beta$ is in fact the only way to proceed here. The obvious problem, then, is how to make the right choice of n, and we now describe a method, essentially due to Hochster [1], to deal with this.

Recall that, on any field, an *additive* (discrete) *valuation* is a map v into $\boldsymbol{Z} \cup \{\infty\}$ such that

$$\begin{aligned} v(xy) &= v(x) + v(y), \\ v(x+y) &= \min\{v(x), v(y)\}, \\ v(x) &= \infty \quad \text{iff} \ \ x = 0, \end{aligned}$$

where $\infty \geq m$ for all $m \in \boldsymbol{Z}$, and the obvious rules for $+$ hold with respect to ∞. In the present context, we consider the valuation v_ξ on F, for each $\xi \in H$, given by the condition

$$v_\xi(z_a) = 1 - \xi(a)$$

so that

$$v_\xi(z_{a_1}^{k_1} \cdots z_{a_n}^{k_n}) = \Sigma k_i(1 - \xi(a_i))$$

and for any

$$u = \Sigma \lambda z_{a_1}^{k_1} \cdots z_{a_n}^{k_n}$$

with distinct monomials and non-zero λ,

$$v_\xi(u) = \min\{\Sigma k_i(1 - \xi(a_i)) \mid \text{ all monomials of } u\}.$$

The notions that will provide a way of dealing with the problem indicated above are now given by

Definition 4. (1) $\alpha \in \Phi$ is called *admissible* if $d(\alpha) \in M$ and, for all $\xi \leq \zeta$ in H, $v_\xi(\alpha(\zeta)) \geq 0$, equality holding iff $\alpha(\xi) \neq 0$.

(2) $\alpha, \beta \in \Phi$ are *compatible* if $d(\alpha) \leq d(\beta)$ and $\alpha \# \beta$ is admissible.

(3) A subring A of Φ is called *admissible* if all $\alpha \in A$ are admissible.

The two separate conditions in (1) will be referred to as the support condition and the valuation condition.

As a first result in this context we now have

Lemma 9. *P is admissible.*

Proof: Since the support condition has already been verified we only have to deal with the valuation condition, which in fact is very easily done: for any $\tau = \Sigma \lambda \sigma_{a_1}^{k_1} \cdots \sigma_{a_n}^{k_n}$ in its reduced representation and any $\xi \leq \zeta$ in H,

$$\begin{aligned} v_\xi(\tau(\zeta)) &= v_\xi\left(\Sigma \lambda \zeta(a_1 \wedge \cdots \wedge a_n) z_{a_1}^{k_1} \cdots z_{a_n}^{k_n}\right) \\ &= \min\{\Sigma k_i(1 - \xi(a_i)) \mid \zeta(a_1 \wedge \cdots \wedge a_n) = 1\} \end{aligned}$$

is always non-negative; further, it is zero iff $\xi(a_1 \wedge \cdots \wedge a_n) = 1$ for some summand of τ (using $\xi \leq \zeta$ for "if"), which in turn holds iff $\tau(\xi) \neq 0$. □

Next, we establish the crucial result that the notions of admissibility and compatibility provide the desired check on the adjunction of conditional quotients.

Lemma 10. *For any admissible subring A of Φ, if $\alpha, \beta \in A$ are compatible, then $A[\alpha\#\beta]$ is again admissible.*

Proof: Consider any $\tau = \alpha_0 + \alpha_1(\alpha\#\beta) + \cdots + \alpha_m(\alpha\#\beta)^m$ in $A[\alpha\#\beta]$. Using Lemma 8 to prove $d(\tau) \in M$, it has to be shown that, for all $\xi \leq \zeta$ in H, $\xi(d(\tau)) \leq \zeta(d(\tau))$, or equivalently, $\xi(d(\tau)) = 0$ whenever $\zeta(d(\tau)) = 0$, which in turn means $\tau(\xi) = 0$ whenever $\tau(\zeta) = 0$. Now, put $\gamma = \tau\beta^m = \alpha_o\beta^m + \alpha_1\alpha\beta^{m-1} + \cdots + \alpha_m\alpha^m$. Then $\tau(\zeta) = 0$ implies $\gamma(\zeta) = 0$, hence also $\gamma(\xi) = 0$ since $\gamma \in A$ so that γ is admissible, and therefore $\tau(\xi) = 0$ or $\beta(\xi) = 0$. Suppose, then, $\tau(\xi) \neq 0$ and consequently $\beta(\xi) = 0$. It follows that $\alpha(\xi) = 0$ since $d(\alpha) \leq d(\beta)$, therefore $\alpha\#\beta(\xi) = 0$, and hence $\tau(\xi) = \alpha_0(\xi)$, showing that $\alpha_0(\xi) \neq 0$. Next, $\gamma(\zeta) = 0$ implies

$$-\alpha_0(\zeta)\beta(\zeta)^m = \alpha_1\alpha\beta^{m-1}(\zeta) + \cdots + \alpha_m\alpha^m(\zeta),$$

and since $\alpha_0(\xi) \neq 0$ we have $v_\xi(\alpha_0(\zeta)) = 0$ by admissibility, which leads to

$$\begin{aligned} mv_\xi(\beta(\zeta)) &\geq \min\{v_\xi(\alpha_i(\zeta)) + iv_\xi(\alpha(\zeta)) + (m-i)v_\xi(\beta(\zeta)) \mid i = 1, \ldots, m\} \\ &\geq kv_\xi(\alpha(\zeta)) + (m-k)v_\xi(\beta(\zeta)) \end{aligned}$$

for some $k \neq 0$, the second step since all $v_\xi(\alpha_i(\zeta)) \geq 0$ by admissibility. With the obvious cancellations we then obtain $v_\xi(\beta(\zeta)) \geq v_\xi(\alpha(\zeta))$ and therefore equality because $\alpha\#\beta$ is admissible. For the same reason, we further have $\alpha\#\beta(\xi) \neq 0$, hence $\alpha(\xi) \neq 0$, therefore also $\beta(\xi) \neq 0$, and consequently $\tau(\xi) = 0$. In all, this shows that $\tau(\xi) = 0$, as desired, proving the support condition for τ. □

To verify the valuation condition for τ, given as above, note that the inequality

$$(\S) \qquad v_\xi(\tau(\zeta)) \geq \min\{v_\xi(\alpha_k(\alpha\#\beta)^k(\zeta)) \mid k = 0, 1, \ldots, n\}$$

shows trivially that $v_\xi(\tau(\zeta)) \geq 0$ since all α_k and $\alpha\#\beta$ are admissible. For equality, we consider two cases:

Case $\beta(\xi) \neq 0$. Then $v_\xi(\beta(\zeta)) = 0$ since β is admissible, therefore $v_\xi(\tau(\zeta)) = v_\xi(\gamma(\zeta))$ since $\gamma = \tau\beta^m$, and hence $v_\xi(\tau(\zeta)) = 0$ iff $v_\xi(\gamma(\zeta)) = 0$ iff $\gamma(\xi) \neq 0$ since γ is admissible, and this holds iff $\tau(\xi) \neq 0$ since $\beta(\xi) \neq 0$.

Case $\beta(\xi) = 0$. Here, $\alpha\#\beta(\xi) = 0$ and hence $\tau(\xi) = \alpha_0(\xi)$. Now, if $\tau(\xi) = 0$, then all terms on the right hand side of (§) are positive and hence $v_\xi(\tau(\zeta)) > 0$. On the other hand, if $\tau(\xi) \neq 0$, then $\alpha_0(\xi) \neq 0$ and therefore $v_\xi(\alpha_0(\zeta)) = 0$, while all other terms involved are positive; by a general property of valuations, this implies $v_\xi(\tau(\zeta)) = 0$.

This lemma provides only one part of what is needed to enforce faithfulness of the standard support by appropriate adjunction; the other part is given by

Lemma 11. *For any admissible $\alpha, \beta \in \Phi$ such that $d(\alpha) \leq d(\beta)$, there exists a natural number m for which α^m, β are compatible.*

PROOF: First, take a natural number k such that $v_\xi(\beta(\zeta)) \leq k$ for all $\xi \leq \zeta$ such that $\beta(\zeta) \neq 0$, and then a natural number m such that $k < mv_\xi(\alpha(\zeta))$ for all $\xi \leq \zeta$ such that $v_\xi(\alpha(\zeta)) > 0$. We claim that α^m, β, are compatible. The support condition is trivial since $d(\alpha^m\#\beta) = d(\alpha^m) = d(\alpha)$ and α is admissible. Concerning the valuation condition, take any $\xi \leq \zeta$ in H. If $\alpha^m\#\beta(\xi) = 0$, then $\alpha(\xi) = 0$ so that $v_\xi(\alpha(\zeta)) > 0$ by admissibility. Now, if $\beta(\zeta) \neq 0$, then $v_\xi(\beta(\zeta)) \leq k < mv_\xi(\alpha(\zeta))$ by the choice of k and m and hence

$$v_\xi(\alpha^m\#\beta(\zeta)) = mv_\xi(\alpha(\zeta)) - v_\xi(\beta(\zeta)) > 0.$$

On the other hand, if $\beta(\zeta) = 0$, then $\alpha^m\#\beta(\zeta) = 0$ and $v_\xi(\alpha^m\#\beta(\zeta)) = \infty > 0$. Further, if $\alpha^m\#\beta(\xi) \neq 0$, then $\alpha(\xi) \neq 0$ and $\beta(\xi) \neq 0$, hence $v_\xi(\alpha(\zeta)) = v_\xi(\beta(\zeta)) = 0$ by admissibility, and therefore $v_\xi(\alpha^m\#\beta(\zeta)) = 0$. □

Definition 5. A subring A of Φ is called *completely admissible* whenever A is admissible, $P \subseteq A$, and $\alpha\#\beta \in A$ for any compatible $\alpha, \beta \in A$.

That this is the type of ring we are looking for is given by

Lemma 12. *For any completely admissible subring A of Φ, the standard support induces an isomorphism $RIdA \to M$.*

PROOF: Since $P \subseteq A$, d maps A onto M. On the other hand, Lemma 11 ensures that, for any $\alpha, \beta \in A$ such that $d(\alpha) \leq d(\beta)$, $\alpha^m\#\beta \in A$ for some m, and therefore $[\alpha] \subseteq [\beta]$. As noted earlier, this makes $d : A \to M$ a perfect support and hence the result, by Lemma 3. □

Finally, it is now obvious how to obtain a completely admissible subring of Φ. For any admissible subring A of Φ let A' be the extension obtained by adjoining all $\alpha\#\beta$ for compatible $\alpha, \beta \in A$. Then A' is the union of the extensions of A by any finite set of these $\alpha\#\beta$, and since each of these is admissible by Lemma 10, A' is again admissible. Further if we define

$$A_0 = A, \quad A_{n+1} = A_n', \quad A^\# = \bigcup A_n,$$

then $A^\#$ is admissible and obviously closed under forming $\alpha\#\beta$ for any compatible $\alpha, \beta \in A^\#$. In particular, $Q = P^\#$ is completely admissible, and we arrive at

Proposition 1. *The standard support induces an isomorphism $RIdQ \to M$.*

Remark. There is a variant of the construction of the above Q for a given finite distributive lattice M which does not start with M itself but with a homomorphism $h : N \to M$ onto M from some other finite distributive lattice. This proceeds as follows: Let $C \supseteq N$ and $B \supseteq M$ be the Boolean envelopes, regarding

h also as the corresponding homomorphism $C \to B$, take F, Φ, P, and Q as defined above, but now for the lattice N, and consider the Boolean power Ψ of F by B. Further, we let $H(N)$ and $H(M)$ be the sets of homomorphism into 2 from N and M respectively. Obviously, each $\alpha \in \Psi$ determines an F-valued function $\widehat{\alpha}$ on $H(M)$ where, as before,

$$\widehat{\alpha}(\xi) = x \quad \text{iff} \quad \xi(\alpha(x)) = 1,$$

for each $\xi \in H(M)$ we have a discrete valuation w_ξ on F determined by

$$w_\xi(z_a) = 1 - \xi(h(a)) \quad (a \in N).$$

The latter can then be used to define $\alpha \in \Psi$ admissible, $\alpha, \beta \in \Psi$ compatible, and subrings $A \subseteq \Psi$ admissible in exact analogy with Definition 4. Further, we have the subring $P^{(h)}$ of Ψ generated by the elements $\sigma_a^{(h)} \in \Psi$ defined by

$$\sigma_a^{(h)}(z_a) = h(a), \quad \sigma_a^{(h)}(0) = \sim h(a) \quad (a \in N)$$

and it is easily seen that the counterparts of Lemmas 9–12 all hold in the present setting. As a result, an iterative adjunction of the appropriate conditional quotients, starting from $P^{(h)}$, leads to a subring $Q^{(h)}$ of Ψ on which the standard support is M-valued perfect.

Now, as observed in Section 2, there is a ring homomorphism $\Phi \to \Psi$ taking α to $\bar{\alpha} = h\alpha$, and since $\bar{\sigma}_a = \sigma_a^{(h)}$, this maps P into $P^{(h)}$. Furthermore, interpreting the elements of Φ and Ψ as F$-$valued functions, we have $\bar{\alpha}(\xi) = \alpha(\xi h)$, for any $\alpha \in \Phi$ and $\xi \in H(M)$, and hence also

$$w_\xi(\bar{\alpha}(\zeta)) = v_{\xi h}(\alpha(\zeta h)),$$

for any $\xi \leq \zeta$ in $H(M)$. This immediately implies that the homomorphism $\Phi \to \Psi$ takes admissible elements in Φ to admissible elements in Ψ, and compatible pairs to compatible pairs (the latter since $\bar{\alpha}\#\bar{\beta} = \overline{\alpha\#\beta}$), so that it induces a homomorphism $Q \to Q^{(h)}$. As a result, we have a commuting square

$$\begin{tikzcd} Q \arrow[r] \arrow[d, "d"'] & Q^{(h)} \arrow[d, "d"] \\ N \arrow[r, "h"'] & M \end{tikzcd}$$

for the standard supports on Q and $Q^{(h)}$, and by Lemma 4 this leads to the commuting square

$$\begin{tikzcd} RIdQ \arrow[r] \arrow[d, "\tilde{d}"'] & RIdQ^{(h)} \arrow[d, "\tilde{d}"] \\ N \arrow[r, "h"'] & M \end{tikzcd}$$

of frame homomorphisms where both $\tilde{d}$ are isomorphisms.

We note in addition that the properties of h as a lattice homomorphism only come into play here when the ring homomorphism $\Phi \to \Psi$, with the resulting $Q \to Q^{(h)}$ is considered. Hence for *any set map* $\varphi : E \to M$ onto M, there is a corresponding ring $Q^{(\varphi)}$, subring of the Boolean power of the rational function field $\boldsymbol{Q}(z_s \,|\, x \in E)$ in indeterminants z_s, $s \in E$, by the Boolean envelope of M, for which the standard support determines an isomorphism $RIdQ^{(\varphi)} \to M$.

4. The general case

We now turn to an arbitrary coherent frame L, letting $B \supseteq K$ be the Boolean envelope of the lattice $K = KL$ of its compact elements, F the rational function field $\boldsymbol{Q}(z_a \,|\, a \in K)$ in indeterminants z_a corresponding to the elements of K, and Φ the Boolean power of F by B. The basic idea here is to view the construction carried out in the previous section, for each finite sublattice of K, as taking place within the one ring Φ, and then to obtain the ring desired here for L as the union of these subrings. In this vein, we introduce the following for any finite sublattice $M \subseteq K$:

its Boolean envelope $B(M) \subseteq B$;

the subfield F_M of F generated by the z_a, $a \in M$;

the subring Φ_M of Φ consisting of all $\alpha \in \Phi$ such that $\alpha(x) \in B(M)$ for each $x \in F$ and $\alpha(x) = 0$ whenever $x \notin F_M$; and

the subring P_M of Φ_M generated by the elements σ_a, $a \in M$, defined in Φ by the same condition as before.

Note that Φ_M is essentially the Boolean power of F_M by $B(M)$, and that the standard support on Φ induces the standard support on Φ_M.

We shall refer to the notions introduced in the previous section for M by a corresponding index or prefix as follows:

For any $\xi \in H(M)$, we have the discrete valuation v_ξ^M on F_M, and we call

$\alpha \in \Phi_M$ *M-admissible* if $d(\alpha) \in M$ and, for all $\xi \leq \zeta$ in $H(M)$, $v_\xi^M(\alpha(\zeta)) \geq 0$, equality holding iff $\alpha(\xi) \neq 0$;

$\alpha, \beta \in \Phi_M$ *M-compatible* if $d(\alpha) \leq d(\beta)$ and $\alpha \# \beta$ is M-admissible;

a subring $A \subseteq \Phi_M$ *M-admissible* if all $\alpha \in A$ are M-admissible.

It is then obvious that, for any finite sublattices M and N of K such that $M \subseteq N$, we have

$$F_M \subseteq F_N, \quad \Phi_M \subseteq \Phi_N, \quad P_M \subseteq P_N.$$

Note that, by the middle inclusion, each $\alpha \in \Phi_M$ determines functions $\widehat{\alpha}_M$ and $\widehat{\alpha}_N$ on $H(M)$ and $H(N)$, respectively. Further, we have the map $H(N) \to H(M)$ by restriction. Next, we show that the relation between the M-concepts and the N-concepts in this situation is as expected.

Lemma 13. (1) *For any* $\xi \in H(N)$, $v^M_{\xi\,|\,M} = v^N_\xi \,|\, F_M$.

(2) *For any* $\alpha \in \Phi_M$ *and* $\xi \in H(N)$, $\widehat{\alpha}_N(\xi) = \widehat{\alpha}_M(\xi\,|\,M)$.

Proof: (1) Obvious.

(2) By definition, $\widehat{\alpha}_N(\xi) = x$ iff $\xi(\alpha(x)) = 1$ while $\widehat{\alpha}_M(\xi\,|\,M) = x$ iff $(\xi\,|\,M)(\alpha(x)) = 1$, and for $\alpha \in \Phi_M$ these are the same conditions.

Corollary. (1) *Any M-admissible $\alpha \in \Phi_M$ is also N-admissible.*

(2) *Any M-compatible $\alpha, \beta \in \Phi_M$ are also N-compatible.*

Proof: (1) For such α, $d(\alpha) \in M$ and, for any $\xi \leq \zeta$ in $H(M)$, $v_\xi(\widehat{\alpha}_M(\zeta)) \geq 0$, equality holding iff $\widehat{\alpha}_M(\xi) \neq 0$. Then also $d(\alpha) \in N$ since $M \subseteq N$, and for any $\xi \leq \zeta$ in $H(N)$, $\xi\,|\,M \leq \zeta\,|\,M$ in $H(M)$ so that

$$v^N_\xi(\widehat{\alpha}_N(\zeta)) = v^M_{\xi\,|\,M}(\widehat{\alpha}_M(\zeta\,|\,M)) \geq 0,$$

equality holding iff $\widehat{\alpha}_M(\xi\,|\,M) \neq 0$ and hence, equivalently, iff $\widehat{\alpha}_N(\xi) \neq 0$. Thus, α is N-admissible.

(2) Obvious, by (1). □

As an immediate consequence of the above we have, with the same notation regarding the adjunction of conditional quotients as in the previous section: For any M-admissible subring $A \subseteq \Phi_M$, and N-admissible subring $B \subseteq \Phi_N$, if $A \subseteq B$, then $A' \subseteq B'$, and consequently also $A^\# \subseteq B^\#$. In particular, $P_M \subseteq P_N$ implies $Q_M \subseteq Q_N$ for $Q_M = P_M^\#$ and $Q_N = P_N^\#$, and since any two finite sublattices of K generate a finite sublattice, the Q_M form an updirected set of subrings of Φ so that their union Q is a subring of Φ. Now, by the properties of the standard supports of the Q_M, it is obvious that, on Q, the standard support is a perfect support $Q \to L$, and hence we have the desired general result:

Proposition 2. *The standard support induces an isomorphism $RIdQ \to L$.*

We close this section with a couple of results concerning the functor RId which correspond to certain results of Hochster [1].

For any coherent frame L, let now Q_L be the ring constructed above and $d_L : Q_L \to L$ the standard support. Then we have

Proposition 3. *For coherent embeddings, the correspondence $L \rightsquigarrow Q_L$ is functorial such that the isomorphisms $\tilde{d}_L : RIdQ_L \to L$ are natural in L.*

Proof: For any coherent frame L, let F_L be the field of rational functions over $\boldsymbol{Q}$ in indeterminates z_a, $a \in KL$, and Φ_L the Boolean power of F_L by $B(KL)$. Then any coherent embedding $h : L \to M$ induces the following:

> a lattice embedding $KL \to KM$, and hence also a lattice embedding $B(KL) \to B(KM)$;
>
> a field embedding $F_L \to F_M$ taking each indeterminate $z_a \in F_L$ to $z_{h(a)} \in F_M$;

a ring embedding $\Phi_L \to \Phi_M$ resulting from (1) and (2), taking $\alpha \in \Phi_L$ to $\bar{\alpha} : F_M \to B(KM)$ which has value 0 outside the image of F_L and sends each y in that image, coming from $x \in F_L$, to the image of $\alpha(x)$ by the embedding $B(KL) \to B(KM)$.

Now, Q_L is defined as the union of certain subrings of Φ_L which, in turn, are determined by the finite sublattices of KL, and it is clear that the embedding $\Phi_L \to \Phi_M$ takes each of these to a subring of Φ_M that occurs in the definition of Q_M and hence provides an embedding $Qh : Q_L \to Q_M$. Obviously, the resulting correspondence $h \rightsquigarrow Qh$ preserves composites and identity maps so that we have the desired functor. The remaining claim results from Lemma 4 and the fact that the diagram

$$\begin{tikzcd} Q_L \arrow[r, "Qh"] \arrow[d, "d_L"'] & Q_M \arrow[d, "d_M"] \\ L \arrow[r, "h"'] & M \end{tikzcd}$$

commutes:

$$\begin{aligned} hd_L(\alpha) &= h\Big(\bigvee\{\alpha(x) \,|\, 0 \neq x \in F_L\}\Big) = \bigvee\{h(\alpha(x)) \,|\, 0 \neq x \in F_L\} \\ &= \bigvee\{\bar{\alpha}(y) \,|\, 0 \neq y \in F_M\} = d_M Qh(\alpha). \end{aligned}$$

Note that an immediate consequence of Proposition 3 is that up to isomorphism, any coherent embedding lies in the image of the functor RId. In actual fact, though, we have

Proposition 4. *Every coherent homomorphism occurs in the image of RId, up to isomorphism.*

Proof: Given any coherent $h : L \to M$, we use the factorization

$$\begin{gathered} L \xrightarrow{\bar{h}} L \times M \xrightarrow{p} M, \\ a \rightsquigarrow (a, h(a)),\ (a, b) \rightsquigarrow B, \end{gathered}$$

noting that both, $\bar{h}$ and p are coherent since $K(L \times M) = KL \times KM$. Here, the embedding $\bar{h}$ is realized by the corresponding $Q\bar{h} : Q_L \to Q_{L\times M}$ in the proof of Proposition 3, and the result will follow if we can produce a ring homomorphism $f : Q_{L\times M} \to A$ which fits into a commuting square

$$\begin{tikzcd} Q_{L\times M} \arrow[r, "f"] \arrow[d, "d"'] & A \arrow[d, "\bar{d}"] \\ L \times M \arrow[r, "p"'] & M \end{tikzcd}$$

where $\bar{d}$ is a perfect support. This can be done as follows: More generally, for any coherent onto homomorphism $h : N \to M$, the finite sublattices M_0 of KM are exactly the images by h of the finite sublattices N_0 of KN, and to each such situation $N_0 \to M_0$ the argument in the Remark following Proposition 1 applies. Combined with the procedure in the proof of Proposition 2, this determines a subring $Q_M^{(h)}$ of the Boolean power of the field F_N by $B(KM)$ such that the standard support $d : Q_M^{(h)} \to M$ is perfect, and the ring homomorphism induced by h between the two Boolean powers involved maps Q_N into $Q_M^{(h)}$. It follows that the commuting square

$$\begin{array}{ccc} Q_N & \longrightarrow & Q_M^{(h)} \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\ N & \xrightarrow[h]{} & M \end{array}$$

is of the desired kind.

Concluding remarks

As mentioned already in the introduction, it is entirely possible to derive the results presented here from the results of Hochster [1] without recourse to PIT, and we now give an outline of the arguments involved.

The following categories enter into this:

$\mathbf{Spec}_{\mathrm{on}}$ – spectral spaces and onto spectral maps,

$\mathbf{FSpec}_{\mathrm{on}}$ – the full subcategory of this given by the finite spectral spaces (which are just the finite T_0 spaces),

$\mathbf{FSpec}$ – finite spectral spaces and all continuous maps,

$\mathbf{FD}$ – finite distributive lattices and their homomorphisms,

$\mathbf{FD}_{\mathrm{mon}}$ – the subcategory of this given by the monomorphisms.

Then, the usual functors Σ and $\mathcal{O}$ induce a dual equivalence

$$\mathbf{FD} \underset{\mathcal{O}}{\overset{\Sigma}{\leftrightarrows}} \mathbf{FSpec},$$

the finite part of Stone Duality, which is valid without any choice principles. This, in turn, restricts to a dual equivalence between $\mathbf{FD}_{\mathrm{mon}}$ and $\mathbf{FSpec}_{\mathrm{on}}$.

Now, it is shown in [1] that the construction of a ring RX with a given spectrum X can be made (contravariantly) functorial on $\mathbf{Spec}_{\mathrm{on}}$ such that the resulting homeomorphisms $\operatorname{Spec} RX \to X$ are natural in X. In particular, this leads to the composite functor

$$S : \mathbf{FD}_{\mathrm{mon}} \xrightarrow{\Sigma} \mathbf{FSpec}_{\mathrm{on}} \xrightarrow{R} \mathbf{Ann}.$$

At this place, it is obviously crucial to know that the arguments in [1] required here are independent of PIT. Now, it is true that [1] makes no distinction between the radical ideal generated by some finite subset B of a ring and the intersection of all prime ideals containing B — which precisely amounts to PIT — but a careful reading shows that using the former notion in the relevant places will work for all the steps needed here.

Next, for any coherent frame L, let $\mathcal{F}L$ be the partially ordered set of all finite sublattices of KL, viewed as a diagram in $\mathbf{FD}_{\mathrm{mon}}$, and put $A = \lim_{\to} S(\mathcal{F}L)$ (the existence of which poses no problem). Since this is an updirected colimit, it follows from Lemma 5 that

$$RIdA = \lim_{\to} RIdS(\mathcal{F}L)$$

in **CohFrm**. On the other hand, the diagram $RIdS(\mathcal{F}L)$ in $\mathbf{FD}_{\mathrm{mon}}$ is equivalent to the original diagram $\mathcal{F}L$ in view of the natural isomorphisms

$$\Sigma RIdX \to \operatorname{Spec} RX \to X$$

in $\mathbf{FSpec}_{\mathrm{on}}$ and the fact that $\mathcal{O}\Sigma \cong Id$ on $\mathbf{FD}_{\mathrm{mon}}$. Finally, by the equivalence between **D** and **CohFrm**, we also have $L = \lim_{\to} \mathcal{F}L$ in **CohFrm**, showing that $RIdA = L$, as desired.

We note that the additional results on functoriality can be obtained along similar lines.

All this obviously amounts to a formidably circuitous route, and it therefore seemed worthwhile to present an argument which is straightforward and selfcontained. Of course,as already mentioned parenthetically, the present results are stronger than the original ones: the natural isomorphisms $\operatorname{Spec} A \cong \Sigma RIdA$ on **Ann** and $\Sigma\mathcal{O}X \cong X$ on **Spec** applied to the former immediately yield the latter.

An alternative, rather different way to obtain Proposition 2 is to circumvent the question of spatiality of the given L by transforming the necessary parts of Hochster's original argument into the topos **E** of sheaves on the *patch frame* of L, that is, the frame of ideals of the Boolean envelope $B(KL)$ which we used in Section 3. One point about this is that the ring Φ we employ here is actually the ring of global sections of a *ring in the topos* **E**, and instead of the subrings of Φ used here in the construction of the ultimately desired ring one can first consider the corresponding rings in **E** which somehow widens the scope. An outline of this approach is presented in Vermeulen [4].

Actually, the version of Hochster's Theorem dealt with there, which aims at *constructive validity* in the sense of topos theory, has a slightly more elaborate formulation: it says that, for any coherent frame L, there exists an algebra A over the rationals such that $RIdA \cong L$ whenever *KL can be indexed by a decidable set E*, that is, there exists an onto map $E \to KL$, where E is decidable. It may be that the approach presented here yields the same result, but that crucially depends on the question whether Lemma 8 is constructively valid. We did not pursue this point.

As a final observation of a very different nature, we note that Proposition 2 offers an alternative proof for the result of Hodges [2] that the Axiom of Choice follows from the Maximal Ideal Theorem for rings: by that proposition, the latter implies its counterpart for bounded distributive lattices, and that, in turn, is well-known to imply the Axiom of Choice. Of course this does not detract from the merits of the direct proof in [2], but it places the result into an illuminating wider context.

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