Vector-valued sequence space $BMC(X)$ and its properties

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Abstract. In this paper, a vector topology is introduced in the vector-valued sequence space $BMC(X)$ and convergence of sequences and sequentially compact sets in $BMC(X)$ are characterized.

Keywords: vector-valued sequence space, topology, series, compact sets

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1. Introduction

When A. Pietsch [4] gave characterizations for nuclearity of locally convex spaces in terms of vector-valued sequence spaces, he introduced a vector-valued sequence space $\ell_1(X)$ with values in a locally convex space X. And when Li Ronglu and Bu Qing-Ying [2] gave characterizations for a locally convex space which contains no copy of c_0, they introduced a vector-valued sequence space $BMC(X)$ with values in a locally convex space X. In fact, $\ell_1(X) = BMC(X)$, the space consisting of bounded multiplier convergent series in X. From [4], [2] it is obviously seen that the space $BMC(X)$ plays an important role in characterizing the structure of spaces in locally convex space theory. It will be seen in [1] that the space $BMC(X)$ also plays an important role in establishing Orlicz-Pettis type theorem for compact operators on locally convex spaces.

In Section 2 of this paper, we introduce a vector topology in the space $BMC(X)$ with values in a topological vector space X and characterize convergence of sequences in $BMC(X)$ and completeness of $BMC(X)$. In Section 3 we consider the space $BMC(X)$ with values in a locally convex space X and characterize sequentially compact subsets of $BMC(X)$ in different ways.

2. Convergence of sequences in $BMC(X)$

In this section, Let X be a separated topological vector space and U_X denote a local base of closed balanced neighbourhoods of 0 in X (see [5]). For a Banach space E, let $B(E)$ denote its closed unit ball. Let

$$BMC(X) = \Big\{\overline{x} = \{x_i\} \in X^{\mathbb{N}} : \text{series } \sum_i t_i x_i \text{ converges for each } \{t_i\} \in \ell_\infty\Big\}.$$

Then $BMC(X)$ is a sequence space with values in X. For a subset A of X, let

$$\widetilde{A} = \Big\{ \overline{x} \in BMC(X) : \sum_{i \geq 1} t_i x_i \in A \ \text{ for each } \ \{t_i\} \in B(\ell_\infty) \Big\}$$

and

$$\widetilde{U}_X = \Big\{ \widetilde{U} : U \in U_X \Big\}.$$

Proposition 2.1. *There is a unique vector topology for $BMC(X)$ for which $\widetilde{U}_X$ is a local base of neighbourhoods of 0. This vector topology will be denoted by τ.*

Proof: By Corollary 3 of [2], for each $\{x_i\} \in BMC(X)$ the set $\{\sum_{i \geq 1} t_i x_i : \{t_i\} \in B(\ell_\infty)\}$ is compact set in X and hence, is bounded. So it follows that $\widetilde{U}$ absorbs each $\overline{x}$ in $BMC(X)$ for each $U \in U_X$. In addition, it is easy to see that $\widetilde{U}$ is balanced for each $U \in U_X$. And for $U, V \in U_X$ such that $U + U \subset V$ it is easy to prove that $\widetilde{U} + \widetilde{U} \subset \widetilde{V}$. Thus we have proved $\widetilde{U}_X$ is an additive filterbase of balanced absorbing subsets of $BMC(X)$. Now, the proof follows from Theorem 5 of [5, p. 45]. □

For a net $\{\overline{x}^{\,\alpha}\}$ in $BMC(X)$, it is easy to see that

$$\tau - \lim_\alpha \overline{x}^{\,\alpha} = 0 \Longleftrightarrow \lim_\alpha \sum_{i \geq 1} t_i x_i^\alpha = 0 \tag{1}$$

uniformly for all $\{t_i\} \in B(\ell_\infty)$. Let

$$P_k : BMC(X) \longrightarrow X, \ P_k(\overline{x}) = x_k;$$

$$I_k : X \longrightarrow BMC(X), \ I_k(x) = (0, \ldots, 0, \overset{(k)}{x}, 0, 0, \ldots).$$

Then P_k and I_k are continuous linear maps, $k = 1, 2, \ldots$.

Lemma 2.2 ([3])**.** *Let $x_{ij} \in X$ for $i, j \in \mathbb{N}$. Suppose*

(I) $\lim_i x_{ij} = x_j$ *exists for each* $j \in \mathbb{N}$ *and*
(II) *for each increasing sequence* $\{m_j\}$ *of* $\mathbb{N}$ *there is a subsequence* $\{n_j\}$ *of* $\{m_j\}$ *such that* $\{\sum_{j \geq 1} x_{in_j}\}_{i=1}^\infty$ *is Cauchy.*

Then $\lim_i x_{ii} = 0$.

Theorem 2.3. *For $\overline{x}^{\,(n)}, \overline{x} \in BMC(X)$, $n = 1, 2, \ldots$, the following statements are equivalent:*

(i) τ-$\lim_n \overline{x}^{\,(n)} = \overline{x}$.
(ii) $\lim_n \sum_{i \geq 1} t_i x_i^{(n)} = \sum_{i \geq 1} t_i x_i$ *for each* $\{t_i\} \in \ell_\infty$.
(iii) $\lim_n x_i^{(n)} = x_i$ *for* $i \in \mathbb{N}$*. And for each* $\{t_i\} \in \ell_\infty$, $\lim_k \sum_{i > k} t_i x_i^{(n)} = 0$ *uniformly for all* $n \in \mathbb{N}$.
(iv) $\lim_n x_i^{(n)} = x_i$ *for* $i \in \mathbb{N}$*. And* $\lim_k \sum_{i > k} t_i x_i^{(n)} = 0$ *uniformly for all* $n \in \mathbb{N}$ *and all* $\{t_i\} \in B(\ell_\infty)$.

Proof: (i) $\Rightarrow$ (iv). By (i), $\lim_n x_i^{(n)} = x_i$ obviously for $i \in \mathbb{N}$. Let $U, V \in U_X$ such that $V + V \subset U$. By (i), there is $n_0 \in \mathbb{N}$ such that for $n \geq n_0$,

$$\sum_{i\geq 1} t_i(x_i^{(n)} - x_i) \in V, \quad \{t_i\} \in B(\ell_\infty).$$

By Example 1 of [2], there is $k_0 \in \mathbb{N}$ such that for $k \geq k_0$ and $n = 1, 2, \ldots, n_0$,

$$\sum_{i>k} t_i x_i^{(n)} \in U, \ \sum_{i>k} t_i x_i \in V, \ \{t_i\} \in B(\ell_\infty). \tag{2}$$

So for $k \geq k_0$ and $n > n_0$,

$$\sum_{i>k} t_i x_i^{(n)} = \sum_{i>k} t_i(x_i^{(n)} - x_i) + \sum_{i>k} t_i x_i \in V + V \subset U, \ \{t_i\} \in B(\ell_\infty). \tag{3}$$

Thus (iv) follows from (2) and (3).

(iv) $\Rightarrow$ (iii). Obviously.

(iii) $\Rightarrow$ (ii). Let $\{t_i\} \in \ell_\infty$, $U, V \in U_X$ such that $V + V + V \subset U$. By (iii), there is $k_0 \in \mathbb{N}$ such that

$$\sum_{i>k_0} t_i x_i^{(n)} \in V, \ \sum_{i>k_0} t_i x_i \in V, \ n = 1, 2, \ldots$$

and there is $n_0 \in \mathbb{N}$ such that for $n > n_0$,

$$\sum_{i=1}^{k_0} t_i(x_i^{(n)} - x_i) \in V.$$

So for $n > n_0$,

$$\sum_{i\geq 1} t_i(x_i^{(n)} - x_i) = \sum_{i=1}^{k_0} t_i(x_i^{(n)} - x_i) + \sum_{i>k_0} t_i x_i^{(n)} - \sum_{i>k_0} t_i x_i \in V + V + V \subset U.$$

(ii) follows.

(ii) $\Rightarrow$ (i). By (ii), $\lim_n x_i^{(n)} = x_i$ obviously for $i \in \mathbb{N}$. If $\tau\text{-}\lim_n \overline{x}^{(n)} \neq \overline{x}$, then there would exist $U \in U_X$, an increasing subsequence $\{n_k\}$ and $\{t_i^{(k)}\} \in B(\ell_\infty)$, $k = 1, 2, \ldots$ such that

$$\sum_{i\geq 1} t_i^{(k)}(x_i^{(n_k)} - x_i) \notin U, \quad k = 1, 2, \ldots .$$

For convenience, we can suppose that

$$\sum_{i\geq 1} t_i^{(n)}(x_i^{(n)} - x_i) \notin U, \quad n = 1, 2, \dots .$$

Let $V, W \in U_X$ such that $V + V \subset W$ and $W + W \subset U$. Pick $m_1 \in \mathbb{N}$ such that $\sum_{i>m_1} t_i^{(1)}(x_i^{(1)} - x_i) \in V$. Then

$$\sum_{i=1}^{m_1} t_i^{(1)}(x_i^{(1)} - x_i) \notin V.$$

Set $n_1 = 1$. Since $\lim_n x_i^{(n)} = x_i$ for $i \in \mathbb{N}$, there is $n_2 \in \mathbb{N}$ with $n_2 > n_1$ such that $\sum_{i=1}^{m_1} s_i(x_i^{(n_2)} - x_i) \in W$ for all $\{s_i\} \in B(\ell_\infty)$. It follows that $\sum_{i=1}^{m_1} t_i^{(n_2)}(x_i^{(n_2)} - x_i) \in W$. So $\sum_{i>m_1} t_i^{(n_2)}(x_i^{(n_2)} - x_i) \notin W$. Pick $m_2 \in \mathbb{N}$ with $m_2 > m_1$ such that $\sum_{i>m_2} t_i^{(n_2)}(x_i^{(n_2)} - x_i) \in V$. Then

$$\sum_{i=m_1+1}^{m_2} t_i^{(n_2)}(x_i^{(n_2)} - x_i) \notin V.$$

Proceeding in this manner we produce increasing sequences $\{n_k\}$ and $\{m_k\}$ such that

$$\sum_{i=m_k+1}^{m_{k+1}} t_i^{(n_{k+1})}(x_i^{(n_{k+1})} - x_i) \notin V, \quad k = 0, 1, 2, \dots, \tag{4}$$

here set $m_0 = 0$. Let

$$y_{kj} = \sum_{i=m_j+1}^{m_{j+1}} t_i^{(n_{j+1})}(x_i^{(n_{k+1})} - x_i).$$

Then $\lim_k y_{kj} = 0$ for $j \in \mathbb{N}$. Set $t_i = t_i^{(n_{j+1})}$ for $m_j < i \leq m_{j+1}$, $j = 0, 1, 2, \dots$, and $t_i = 0$ elsewhere. Then $\{t_i\} \in \ell_\infty$ and $\sum_{j\geq 0} y_{kj} = \sum_{i\geq 1} t_i(x_i^{(n_{k+1})} - x_i)$. By (ii), $\lim_k \sum_{j\geq 0} y_{kj} = 0$. So it follows from Lemma 2.2 that $\lim_k y_{kk} = 0$. This contradicts (4) and (i) follows.

The proof of Theorem 2.3 is complete. □

Proposition 2.4. *$BMC(X)$ is complete (or sequentially complete) space if and only if X is complete (or sequentially complete) space.*

Proof: If $BMC(X)$ is complete space, then it is easy to prove that X is complete. Conversely, if X is complete space, we will prove that $BMC(X)$ is complete space.

Let $\{\overline{x}^{\,\alpha}\}$ be Cauchy net in $BMC(X)$ and $U, V \in U_X$ such that $V+V+V \subset U$. Then for $\widetilde{V} \in \widetilde{U}_X$ there is α_0 such that for $\alpha, \beta \geq \alpha_0$, $\overline{x}^{\,\alpha} - \overline{x}^{\,\beta} \in \widetilde{V}$, i.e. for $\alpha, \beta \geq \alpha_0$,

$$\sum_{i \geq 1} t_i (x_i^\alpha - x_i^\beta) \in V, \quad \{t_i\} \in B(\ell_\infty). \tag{5}$$

By the continuity of P_i, $\{x_i^\alpha\}$ is Cauchy net in X and hence, there is $x_i \in X$ such that

$$\lim_\alpha x_i^\alpha = x_i, \quad i = 1, 2, \ldots\ . \tag{6}$$

From (5) it follows that for $\alpha, \beta \geq \alpha_0$ and each $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} t_i (x_i^\alpha - x_i^\beta) \in V, \quad \{t_i\} \in B(\ell_\infty).$$

So by (6) for $\alpha \geq \alpha_0$ and $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} t_i (x_i^\alpha - x_i) \in V, \quad \{t_i\} \in B(\ell_\infty).$$

Because of Example 1 of [2], there is $n_0 \in \mathbb{N}$ such that for $n > n_0$,

$$\sum_{i>n} t_i x_i^{\alpha_0} \in V, \quad \{t_i\} \in B(\ell_\infty).$$

Thus for $n > n_0$ and $\alpha \geq \alpha_0$,

$$\begin{aligned}\sum_{i=1}^{n} t_i x_i - \sum_{i \geq 1} t_i x_i^\alpha = \sum_{i=1}^{n} t_i (x_i - x_i^\alpha) - \sum_{i>n} t_i (x_i^\alpha - x_i^{\alpha_0}) - \\ - \sum_{i>n} t_i x_i^{\alpha_0} \in V + V + V \subset U, \quad \{t_i\} \in B(\ell_\infty).\end{aligned}$$

It follows that the series $\sum_i t_i x_i$ converges for each $\{t_i\} \in \ell_\infty$, i.e. $\overline{x} = \{x_i\} \in BMC(X)$ and for $\alpha > \alpha_0$,

$$\sum_{i \geq 1} t_i x_i - \sum_{i \geq 1} t_i x_i^\alpha \in U, \quad \{t_i\} \in B(\ell_\infty).$$

So $\tau\text{-}\lim_\alpha \overline{x}^{\,\alpha} = \overline{x}$ and we have proved that $BMC(X)$ is complete. The proof is complete. $\square$

3. Compact sets in $BMC(X)$

In this section, let X be a locally convex space and X' its dual space. Then (X, X') forms a dual pair. Let U_X denote a local base of barrelled neighbourhoods of 0 in X. The gauge of $U \in U_X$ will be denoted by p_U and the polar of U will be denoted by U^0 (see [5]). It is easy to see that

$$p_U(x) = \sup\{|f(x)| : f \in U^0\}, \quad x \in X. \tag{7}$$

For each $U \in U_X$ and each $\overline{x} = \{x_i\} \in BMC(X)$, let

$$\varepsilon_U(\overline{x}) = \sup\Big\{p_U\Big(\sum_{i\geq 1} t_i x_i\Big) : \{t_i\} \in B(\ell_\infty)\Big\}. \tag{8}$$

Then $\varepsilon_U(\cdot)$ is a seminorm on $BMC(X)$ and the topology generated by the family of seminorms $\{\varepsilon_U(\cdot) : U \in U_X\}$ on $BMC(X)$ is just the original topology τ.

Proposition 3.1. *For each $U \in U_X$ and each $\overline{x} \in BMC(X)$,*

$$\varepsilon_U(\overline{x}) = \sup\Big\{\sum_{i\geq 1} |f(x_i)| : f \in U^0\Big\}. \tag{9}$$

The proof follows from (7) and (8).

For $t = \{t_i\} \in \ell_\infty$, let

$$\varphi_t : BMC(X) \longrightarrow X, \quad \varphi_t(\overline{x}) = \sum_{i\geq 1} t_i x_i.$$

Then for each $U \in U_X$, $p_U(\varphi_t(\overline{x})) \leq \varepsilon_U(\overline{x})$. So φ_t is a continuous linear map.

By Example 1 of [2], it is known that each $\{x_i\} \in BMC(X)$ has the following property:

$$\tau\text{-}\lim_n \sum_{i>n} I_i(x_i) = 0. \tag{$*$}$$

In order to consider a subset of $BMC(X)$, we give

Definition 3.2. A subset A of $BMC(X)$ is called uniformly convergent if τ-$\lim_n \sum_{i>n} I_i(x_i) = 0$ uniformly for all $\{x_i\} \in A$; A is called $\sigma(X, X')$-uniformly convergent if for each $f \in X'$, $\lim_n \sum_{i>n} |f(x_i)| = 0$ uniformly for all $\{x_i\} \in A$.

Theorem 3.3. *Let X be a sequentially complete space and A a subset of $BMC(X)$. Then A is relatively sequentially compact if and only if*

(a) *A is uniformly convergent and*
(b) *for each $i \in \mathbb{N}$, $P_i(A)$ is relatively sequentially compact subset of X.*

Proof: If A is relatively sequentially compact, then (b) holds obviously. Next we will prove that (a) holds.

Suppose that (a) does not hold. Then there is $U \in U_X$ such that

$$\limsup_n \Big\{\varepsilon_U\Big(\sum_{i>n} I_i(x_i)\Big) : \overline{x} = \{x_i\} \in A\Big\} \neq 0,$$

i.e.

$$\limsup_n \Big\{p_U\Big(\sum_{i>n} t_i x_i\Big) : \{t_i\} \in B(\ell_\infty),\ \overline{x} = \{x_i\} \in A\Big\} \neq 0.$$

So there are $\varepsilon_0 > 0$, increasing subsequence $\{n_k\}$ of $\mathbb{N}$, $\{t_i^{(k)}\} \in B(\ell_\infty)$ and $\overline{x}^{(k)} \in A$ such that

$$p_U\Big(\sum_{i>n_k} t_i^{(k)} x_i^{(k)}\Big) \geq \varepsilon_0, \quad k = 1, 2, \dots . \tag{10}$$

Since A is relatively sequentially compact, there are a subsequence $\{\overline{x}^{(k_j)}\}_1^\infty$ of $\{\overline{x}^{(k)}\}_1^\infty$ and $\overline{x} \in BMC(X)$ such that $\tau\text{-}\lim_j \overline{x}^{(k_j)} = \overline{x}$. By Theorem 2.3,

$$\limsup_m \Big\{p_U\Big(\sum_{i>m} t_i x_i^{(k_j)}\Big) : \{t_i\} \in B(\ell_\infty),\ j \in \mathbb{N}\Big\} = 0.$$

So there is n_{k_j} such that

$$p_U\Big(\sum_{i>n_{k_j}} t_i^{(k_j)} x_i^{(k_j)}\Big) < \varepsilon_0.$$

This contradicts (10). Thus we have proved that (a) holds.

Conversely, if the conditions (a) and (b) hold, we will prove that A is relatively sequentially compact. Let $\{\overline{x}^{(n)}\}_1^\infty \subset A$. By (b), using the diagonal method we can find a subsequence $\{n_k\}$ of $\mathbb{N}$ such that $\lim_k x_i^{(n_k)}$ exists for $i \in \mathbb{N}$. For convenience, we can suppose that $n_k = k$, i.e.

$$\lim_n x_i^{(n)} \text{ exists}, \quad i = 1, 2, \dots . \tag{11}$$

By (a) for each $U \in U_X$ and $\varepsilon > 0$, there is $k_0 \in \mathbb{N}$ such that

$$\varepsilon_U\Big(\sum_{i>k_0} I_i(x_i)\Big) < \varepsilon/4 \text{ for } \{x_i\} \in A.$$

And furthermore, by (11) there is $n_0 \in \mathbb{N}$ such that for $n, m > n_0$,

$$p_U\big(x_i^{(n)} - x_i^{(m)}\big) < \varepsilon/2k_0, \quad i = 1, 2, \ldots, k_0.$$

Thus for $n, m > n_0$,

$$\varepsilon_U\big(\overline{x}^{(n)} - \overline{x}^{(m)}\big) \le \sum_{i=1}^{k_0} p_U\big(x_i^{(n)} - x_i^{(m)}\big) + \varepsilon_U\Big(\sum_{i>k_0} I_i(x_i^{(n)})\Big) + \varepsilon_U\Big(\sum_{i>k_0} I_i(x_i^{(m)})\Big) < \varepsilon.$$

So $\{\overline{x}^{(n)}\}_1^\infty$ is a Cauchy sequence of $BMC(X)$ and hence, $\tau\text{-}\lim_n \overline{x}^{(n)}$ exists in $BMC(X)$ by Proposition 2.4. Thus we have proved that A is relatively sequentially compact. The proof is complete.

□

Lemma 3.4. *For each $t = \{t_i\} \in \ell_\infty$, φ_t is c.c.t. – $\sigma(X, X')$ continuous on each $\sigma(X, X')$-uniformly convergent subset of $BMC(X)$, where c.c.t. denotes the coordinatewise convergence topology on $BMC(X)$.*

Proof: Let A be an $\sigma(X, X')$-uniformly convergent subset of $BMC(X)$ and $\{\overline{x}^\alpha\}$ be a net of A such that $\lim_\alpha x_i^\alpha = 0$ for $i \in \mathbb{N}$. Thus for $\varepsilon > 0$ and $f \in X'$, there is $n_0 \in \mathbb{N}$ such that

$$\sum_{i>n_0} |f(x_i)| < \varepsilon/2, \quad \text{for } \ \overline{x} = \{x_i\} \in A.$$

And hence, there is α_0 such that for $\alpha > \alpha_0$,

$$|f(x_i^\alpha)| < \varepsilon/2n_0, \quad i = 1, 2, \ldots, n_0.$$

So for $\alpha > \alpha_0$,

$$|f(\varphi_t(\overline{x}^\alpha))| \le \sum_{i=1}^{n_0} |f(x_i^\alpha)| + \sum_{i>n_0} |f(x_i^\alpha)| < \varepsilon.$$

Thus we have proved that $\sigma(X, X') - \lim_\alpha \varphi_t(\overline{x}^\alpha) = 0$. The proof is complete.

□

Theorem 3.5. *Let X be a sequentially complete space which contains no copy of c_0. Then a subset A of $BMC(X)$ is relatively sequentially compact if and only if*

(c) *A is $\sigma(X, X')$-uniformly convergent and*
(d) *for each $t \in \ell_\infty$, $\varphi_t(A)$ is relatively sequentially compact subset of X.*

Proof: If A is relatively sequentially compact, then by the continuity of φ_t and Theorem 3.3, the conditions (c) and (d) hold.

Conversely, if the conditions (c) and (d) hold, we will prove that A is relatively sequentially compact. Let $\{\overline{x}^{(n)}\}_1^\infty \subset A$. By use of the proof of Theorem 3.3, we can suppose that

$$\lim_n x_i^{(n)} = x_i^{(0)} \in X, \quad i = 1, 2, \dots . \tag{12}$$

Next we will prove that $\overline{x}^{(0)} = \{x_i^{(0)}\} \in BMC(X)$.

For $f \in X'$, by (c) there is $k_0 \in \mathbb{N}$ such that $\sum_{i>k_0} |f(x_i)| \leq 1$ for each $\overline{x} \in A$. Since (d) implies (b), $\bigcup_{i=1}^{k_0} P_i(A)$ is a relatively sequentially compact subset of X and hence bounded. So there is a constant $c > 0$ such that

$$|f(P_i(\overline{x}))| = |f(x_i)| \leq c, \ \overline{x} = \{x_i\} \in A, \ i = 1, 2, \dots, k_0.$$

Thus

$$\sum_{i\geq 1} |f(x_i)| \leq k_0 c + 1, \quad \overline{x} = \{x_i\} \in A.$$

Now for a fixed $m \in \mathbb{N}$, by (12) there is an $n_0 \in \mathbb{N}$ such that

$$|f(x_i^{(n_0)} - x_i^{(0)})| < 1/m, \quad i = 1, 2, \dots, m.$$

So

$$\sum_{i=1}^m |f(x_i^{(0)})| \leq \sum_{i=1}^m |f(x_i^{(n_0)} - x_i^{(0)})| + \sum_{i=1}^m |f(x_i^{(n_0)})| \leq k_0 c + 2.$$

Since $m \in \mathbb{N}$ is arbitrary, we have $\sum_{i\geq 1} |f(x_i^{(0)})| \leq k_0 c + 2 < \infty$. Therefore, the series $\sum_i x_i^{(0)}$ is a weakly unconditionally Cauchy series in X. It follows from Theorem 4 of [2] that the series $\sum_i x_i^{(0)}$ is unconditionally convergent and hence bounded multiplier convergent. Thus we have proved that $\overline{x}^{(0)} \in BMC(X)$.

Now let $D = A \cup \{\overline{x}^{(0)}\}$. For each $t = \{t_i\} \in \ell_\infty$, since Lemma 3.4 implies that φ_t is c.c.t. $-\,\sigma(X, X')$ continuous on D, by (12) we have $\sigma(X, X')-\lim_n \varphi_t(\overline{x}^{(n)}) = \varphi_t(\overline{x}^{(0)})$. By use of the condition (d), we have $\lim_n \varphi_t(\overline{x}^{(n)}) = \varphi_t(\overline{x}^{(0)})$, i.e. $\lim_n \sum_{i\geq 1} t_i x_i^{(n)} = \sum_{i\geq 1} t_i x_i^{(0)}$. It follows from Theorem 2.3 that $\tau\text{-}\lim \overline{x}^{(n)} = \overline{x}^{(0)}$. So we have proved that A is relatively sequentially compact. The proof is complete. □

Remark 3.6. Condition (d) in Theorem 3.5 cannot be replaced by condition (b) in Theorem 3.3. For example, let $X = \ell_p$ $(1 < p < \infty)$, $e_i = (0, \dots, 0, \overset{(i)}{1}, 0, 0, \dots)$ and $A = \{(0, \dots, 0, e_n, 0, 0, \dots)\}_1^\infty$. Then $A \subset BMC(X)$ and it is easy to prove that A satisfies the conditions (b) and (c) but does not satisfy (a) and (d), and so is not relatively sequentially compact.

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