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Set valued measures and integral representation

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Abstract. The extension theorem of bounded, weakly compact, convex set valued and weakly countably additive measures is established through a discussion of convexity, compactness and existence of selection of the set valued measures; meanwhile, a characterization is obtained for continuous, weakly compact and convex set valued measures which can be represented by Pettis-Aumann-type integral.

Keywords: set valued functions, set valued measures, Pettis-Aumann integral
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Z. Artstein ([2], 1972) introduced the concept of set valued measure in R^n and studied its convexity, the existence of selection and the Radon-Nikodym Property (RNP, for simplicity) corresponding to the Aumann integral (defined by Aumann [1], see also [6], [9] for further properties). Hiai [7] generalized Artstein's results to bounded variation set valued measures on Banach spaces. In 1985, Papageorgiou [10] studied the representation of set valued operators and later on ([11], 1987) he paid attention to distribution theory of set valued functions and measures.

In the present paper, as a generalization and development of Artstein's, Hiai's and Papageorgiou's work, the extension theorem (§3) of bounded, weakly compact, convex set valued and weakly countably additive measures and a characterization (§4) of continuous, weakly compact and convex set valued measures which, defined on a complete and finite measure space, can be represented by Pettis-Aumann type integral are given.

Notation. The letter X will always denote a real Banach space, X' its dual, $\langle\cdot,\cdot\rangle$ the bilinear conjugate operation. $P_a(X)$ is for the set consisting of all nonempty subsets of X and $P_{wee}(X)$ ($\subset P_a(X)$) for all of weakly compact convex subset of X. The symbol "$\to$" ("$\xrightarrow{w}$" and "$\xrightarrow{w^*}$") means to be "norm convergent to" ("weakly convergent to" and "weakly $*$ convergent to", respectively). For $A \subset X$, $\mathrm{co}(A)$ ($\overline{\mathrm{co}}(A)$) denotes the (norm closed) convex hull of A and $\mathrm{cl}(A)$ ($\mathrm{cl}^w(A)$) stands for the norm (weak) closure of A; σ_A, defined by $\sigma_A(x') = \sup\{\langle x', x\rangle : x \in A\}$ for $x' \in X'$, is called the support function of A. The symbol H denotes the Hausdorff metric, that is, $H(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)$, where the metric d is deduced by the norm$\}$, in particular, $H(A,0)$ is denoted by $|A|$ for simplicity. Ω is always a nonempty set, $\mathcal{F}$ and Σ an algebra and σ-algebra, respectively, both $\mathcal{F}$ and Σ consist of subsets of Ω.

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1. Properties of set valued measure

Let $M : \mathcal{F} \to P_a(X)$ be a mapping.

Definition 1.1. (a) M is called finitely additive (set valued) measure on $\mathcal{F}$, if it satisfies (i) $M(\emptyset) = 0$ and (ii) $M(E_1 \cup E_2) = M(E_1) + M(E_2)$ for all $E_1, E_2 \in \mathcal{F}$ with $E_1 \cap E_2 = \emptyset$;

(b) M is said to be countably additive measure provided it is finitely additive and for any mutually disjoint sequence $\{E_n\} \subset \mathcal{F}$, $M(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} M(E_n)$ $= \{x \in X$; for each positive integer n, there is $x_n \in M(E_n)$ such that $\sum_{n=1}^{\infty} x_n$ unconditionally converges to $x\}$.

(c) We say M is weakly countably additive provided for any $x' \in X'$, $\sigma_{M(\cdot)}(x')$ is a real valued measure on $\mathcal{F}$.

(d) We call M bounded provided there exists $C \in R^+$ such that $|M(E)| \equiv \{\|x\|; x \in M(E)\} \leq C$ for all $E \in \mathcal{F}$.

(e) Let $\{A_n\} \subset P_a(X)$. $\sum_{n=1}^{\infty} A_n$ is said to be unconditionally convergent if $\forall\, x_n \in A_n$, for $n = 1, 2, \dots$, $\sum_{n=1}^{\infty} x_n$ is an unconditionally convergent series.

Definition 1.2. M is called strongly additive provided it is finitely additive and $\sum_{n=1}^{\infty} M(E_n)$ unconditionally converges for any mutually disjoint sequence $\{E_n\} \subset \mathcal{F}$.

Lemma 1.3. *Let $\{A_n\}$ ($\subset P_a(X)$) be a uniformly bounded set sequence. If for any subsequence $\{A_{n_k}\} \subset \{A_n\}$ there exists a weakly relatively compact set $A \subset X$ such that $\sum_{n=1}^{\infty} \sigma_{A_{n_k}}(x') \leq \sigma_A(x')$ for all $x' \in X'$, then $\sum_{n=1}^{\infty} A_n$ unconditionally converges.*

Proof: Clearly, $\sum_{n=1}^{\infty} |\sigma_{A_n}(x')| < \infty$ for all $x' \in X'$. For any $x_n \in A_n$, $n = 1, 2, \dots$ and positive integer m, we have

$$\sum_{n=1}^{m} |\langle x', x_n\rangle| \leq \sum_{n=1}^{\infty} |\sigma_{A_n}(x')| + \sum_{n=1}^{\infty} |\sigma_{A_n}(-x')|.$$

Therefore $\sum_{n=1}^{\infty} |\langle x', x_n\rangle| < \infty$. By the Orlicz-Pettis theorem (see, for instance, [5]), it suffices to show that for any sequence $\{x_{n_k}\}$ with $x_{n_k} \in A_{n_k}$, $\sum_k x_{n_k}$ weakly converges. Set $y_m = \sum_{n=1}^{m} x_{n_k}$, we know that $\mathrm{SUP}_m |\langle x', y_m\rangle| < \infty$ for any $x' \in X'$. This and the Resonance Theorem imply that there exists $y \in X''$ such that $y_m \xrightarrow{w^*} y$ in X'', we claim that $y \in X$. Since $\langle y, x'\rangle = \langle \sum_{k=1}^{\infty} x_{n_k}, x'\rangle =$ $\sum_{k=1}^{\infty} \langle x', x_{n_k}\rangle \leq \sum_{k=1}^{\infty} \sigma_{A_{n_k}}(x') \leq \sigma_A(x')$ for some weakly relatively compact $A \in P_a(X)$ and for all $x' \in X'$, that is, y is continuous according to the Makey's topology by $\sigma_A(x')$, we obtain $y \in X$, and this says that $y_m \xrightarrow{w} y$ in X. Thus the proof is complete. $\square$

Corollary 1.4. *Let $M(\Sigma \to P_{wee}(X))$ be a bounded countably additive set valued measure, then M is strongly additive.*

Proof: Let $\{E_n\} \subset \Sigma$ be any disjoint sequence. Set $E = \bigcup_{n=1}^{\infty} E_n$ $(\in \Sigma)$, then $M(E)$ is weakly compact and convex set satisfying $\sigma_{\sum_{n=1}^{\infty} M(E_n)}(x') = \sigma_{M(E)}(x')$ by the countable additivity of M. It is easy to observe that $\sigma_{\sum_n M(E_n)}(x') = \sum_{n=1}^{\infty} \sigma_{M(E_n)}(x')$ for all $x' \in X'$. And Lemma 1.3 implies that $\sum_{n=1}^{\infty} M(E_n)$ is unconditionally convergent. $\square$

Corollary 1.5. *Let $M(\Sigma \to P_{wee}(X))$ be a bounded mapping. Then M is countably additive if and only if M is weakly countably additive.*

Definition 1.6. Let $M(\mathcal{F} \to P_a(X))$ be bounded and finitely additive, we say M is of σ-bounded variation (set valued) mapping, if there exists an $\mathcal{F}$-partition $\{E_n\}$ of Ω such that

$$|M|(E_n) \equiv \mathrm{SUP}_{\Pi_n} \sum_{A \in \Pi_n} |M(A)| < \infty \quad \text{for} \quad n = 1, 2, \ldots$$

where Π_n denotes any finite $\mathcal{F}$-partition of E_n.

For $A \subset \mathcal{F}$ with $M(A) \neq \{0\}$, A is called an atom of M, if either $M(B) = \{0\}$ or $M(A \setminus B) = \{0\}$ whenever $B \in \mathcal{F}$ with $B \subset A$. We say M is non-atomic if M has no atom.

Proposition 1.7. *Let X posses the Radon-Nikodym Property (RNP) and let $M(\Sigma \to P_a(X))$ be a countably additive, non-atomic and σ-bounded variation mapping, then $\operatorname{cl} M(E)$ is convex in X for all $E \in \Sigma$.*

Proof: Suppose that $\{E_n\}$ is a Σ-partition of Ω with $|M|(E_n) < \infty$ for $n = 1, 2, \ldots$, then, for any $E \in \Sigma$, $\operatorname{cl} M(E \cap E_n)$ is a convex set (see, for instance, Hiai [7, Theorem 1.2]). The convexity of the set $\operatorname{cl} M(E)$ will be proved if we show that for any $\varepsilon > 0$, $x_j \in M(E)$ for $j = 1, 2$, and $\alpha \in (0, 1)$, there exists $x \in M(E)$ such that $\|\alpha x_1 + (1 - \alpha) x_2 - x\| < \varepsilon$.

Since $x_j \in M(E) = \sum_{n=1}^{\infty} M(E \cap E_n)$ for $j = 1, 2$, there must be $\{x_n^{(j)}\} \subset X$ with $x_n^{(j)} \in M(E \cap E_n)$ for $n = 1, 2, \ldots$ such that $x_j = \sum_{n=1}^{\infty} x_n^{(j)}$ is unconditionally convergent for $j = 1, 2$. For any fixed $\varepsilon > 0$ and for each positive integer n, there is $x_{(n,\varepsilon)} \in M(E \cap E_n)$ satisfying $\|\alpha x_n^{(1)} + (1 - \alpha) x_n^{(2)} - x_{(n,\varepsilon)}\| < \frac{\varepsilon}{2^n}$. Next we prove that $\sum_{n=1}^{\infty} x_{(n,\varepsilon)}$ is unconditionally convergent series. For all $\delta > 0$, choose a positive integer N such that $\|\sum_{n=m}^{\infty} \varepsilon_n x_n^{(j)}\| < \delta$ for $j = 1, 2$, whenever $m \geq N$, where $\varepsilon_i = \pm 1$ for $i = 1, 2, \ldots$. Further,

$$\begin{aligned}\|\sum_{n=m}^{m+k} \varepsilon_n x_{(n,\varepsilon)}\| \leq \alpha \|\sum_{n=m}^{m+k} \varepsilon_n x_n^{(1)}\| + (1 - \alpha) \|\sum_{n=m}^{m+k} \varepsilon_n x_n^{(2)}\| + \\ + \|\sum_{n=m}^{m+k} \varepsilon_n [x_{(n,\varepsilon)} - (\alpha x_n^{(1)} + (1 - \alpha) x_n^{(2)})]\| \leq \delta + 2^{-m+1} \varepsilon\end{aligned}$$

for any integer $k \geq 0$ and $m \geq N$. This explains that $\sum_n x_{(n,\varepsilon)}$ $(\in M(E))$ is unconditionally convergent. It is easy to see that $\|\alpha x_1+(1-\alpha)x_2-\sum_n x_{(n,\varepsilon)}\| < \varepsilon$, which completes our proof. □

Proposition 1.8. *Let $M(\Sigma \to P_a(X))$ be bounded, countably additive and weakly relatively compact valued. Then $\overline{\text{co}}M$ is also countably additive.*

PROOF: Let $\{E_n\}$ $(\subset \Sigma)$ be any mutually disjoint set sequence, then from $\sigma_{M(\bigcup_{n=1}^\infty E_n)}(x') = \sum \sigma_{M(E_n)}(x')$ it follows that $\sigma_{\overline{\text{co}}M(\bigcup_{n=1}^\infty E_n)}(x') = \sum_{n=1}^\infty \sigma_{\overline{\text{co}}M(E_n)}(x')$. So $\overline{\text{co}}M(\cdot)$ is weakly countably additive. This and Corollary 2.5 imply that $\overline{\text{co}}M(\cdot)$ is countably additive.

□

2. Compactness of set valued measure

Let $M(\Sigma \to P_a(X))$ be a countably additive measure, m is said to be a selection of M provided m is a single X-valued measure satisfying $m(E) \in M(E)$ for any $E \in \Sigma$. For $A \subset X$, we say that x $(\in A)$ is exposed point of A if there exists $x' \in X'$ such that $\langle x', x\rangle > \langle x', y\rangle$ whenever $y \in A \setminus \{x\}$.

Lemma 2.1. *Suppose that $(\Sigma \to P_{wee}(X))$ is countably additive and suppose that x is an exposed point of $M(\Omega)$. Then there exists a selection m of M such that $m(\Omega) = x$.*

PROOF: Suppose that $x' \in X'$ satisfies $\langle x', x\rangle > \langle x', y\rangle$ for all $y \in M(\Omega) \setminus \{x\}$. Since $M(\Omega) = M(E) + M(\Omega \setminus E)$ for any $E \in \Sigma$, there exist u and v with $u \in M(E)$ and $v \in M(\Omega \setminus E)$ such that $x = u + v$. Since

$$\langle x', y\rangle + \langle x', v\rangle = \langle x', x\rangle > \langle x', w\rangle + \langle x', v\rangle$$

for all $w \in M(E) \setminus \{u\}$, we have

$$\langle x', u\rangle > \langle x', w\rangle \quad \text{for} \quad w \in M(E) \setminus \{u\}, \tag{2.1}$$

that is, u is an exposed point of $M(E)$. We denote by $u(E, x')$ the unique exposed point of $M(E)$ satisfying inequality (2.1) and define $m(\Sigma \to X)$ by letting $m(E) = u(E, x')$, then $\sigma_{m(E)}(x') = \sum_{m(E)}(x')$ for all $E \in \Sigma$. It remains to show that m is a single valued measure.

Let $\{E_n\} \subset \Sigma$ be any mutually disjoint sequence, since $M(\bigcup E_n)$ $(= \sum_{n=1}^\infty M(E_n))$ is convex and weakly compact, by Corollary 1.4, $\sum_{n=1}^\infty M(E_n)$ unconditionally converges. This further implies that $\sum_{n=1}^\infty m(E_n)$ is unconditionally convergent to a point of $M(\bigcup E_n)$. Note

$$\sigma_{\Sigma m(E_n)}(x') = \sum_{n=1}^\infty \langle x', m(E_n)\rangle = \sum_{n=1}^\infty \sigma_{M(E_n)}(x') = \sigma_{\sum_{n=1}^\infty M(E_n)}(x') =$$

$$= \sigma_{M(\bigcup E_n)}(x') = \langle x', m(\bigcup E_n)\rangle$$

and note that both $\sum_{n=1}^{\infty} m(E_n)$ and $m(\bigcup_{n=1}^{\infty} E_n)$ are in $M(\bigcup_{n=1}^{\infty} E_n)$, by the uniqueness of $y \in M(\bigcup_{n=1}^{\infty} E_n)$ satisfying $\langle x', y\rangle = \sigma_{M(\bigcup E_n)}(x')$ we obtain that $\sum_{n=1}^{\infty} m(E_n) = m(\bigcup_{n=1}^{\infty} E_n)$, which completes our proof. □

Theorem 2.2. *Suppose that $M(\Sigma \to P_{wee}(X))$ is bounded and countably additive. Then for any $E \in \Sigma$ and $x \in M(E)$ there exists a selection m of M such that $m(E) = x$.*

Proof: Without loss of generality we can assume that $E = \Omega$. Since $M(\Omega)$ is convex and weakly compact, it must be the norm closed convex hull of its exposed points (see, for instance, Amir and Lindenstrauss [3]). Let Q denote the set of all exposed points of $M(\Omega)$. Then for any $x \in M(\Omega)$ and $\varepsilon > 0$ there exist $x_j \in Q$ and $\alpha_j > 0$ for $j = 1, 2, \ldots, n$ with $\sum_{j=1}^{n} \alpha_j = 1$ such that $\|x - \sum_{j=1}^{n} \alpha_j x_j\| < \varepsilon$. Lemma 2.1 implies that there exist selections m_j of M, satisfying $m_j(\Omega) = x_j$ for $j = 1, 2, \ldots, n$. Define $\overline{m}(\Sigma \to X)$ by $\overline{m}(E) = \sum_{j=1}^{n} \alpha_j m_j(E)$, it is easy to observe that $\overline{m}$ is also a selection of M with $\|\overline{m}(\Omega) - x\| < \varepsilon$. In particular, we obtain a selection sequence $\{m^{(k)}\}$ such that $\|m^{(k)}(\Omega) - x\| \to 0$ by letting $\varepsilon = \frac{1}{k}$ for $k = 1, 2, \ldots$. Now we consider the product space $\prod \equiv \prod_{E\in\Sigma} M(E)$ which is equipped with the product topology and $M(E)$ with the weak topology for all $E \in \Sigma$. The sequence $\{\prod_{E\in\Sigma} m^{(k)}(E)\}$ in $\prod$ has a subnet, which is still denoted by $\{\prod_{E\in\Sigma} m^{(k)}(E)\}$ for simplicity, converging to some point $\prod_{E\in\Sigma} m^{(\infty)}(E)$ in $\prod$, since $\prod$ is compact according to the product topology. This explains that $m^{(k)}(E) \to m^{(\infty)}(E) \in M(E)$ for any $E \in \Sigma$ and $m^{(\infty)}(\Omega) = x$. It remains to show that $m^{(\infty)}$ is an X-valued measure.

Assume that $\{E_n\}$ is a mutually disjoint sequence in Σ. By a simple argument we know $\sum_{n=1}^{\infty} |\sigma_{M(E_n)}(x')| < \infty$ for all $x' \in X'$. For any $\varepsilon > 0$ and for any fixed $x \in X'$, choose a positive integer n_0 such that $\sum_{j>n_0} |\sigma_{M(E_j)}(x')| < \frac{\varepsilon}{2}$, then we have $(n \geq n_0)$

$$|\langle x', m^{(k)}(\bigcup_{n=1}^{\infty} E_n) - \sum_{j=1}^{n} m^{(k)}(E_j)\rangle| = |\langle x', \sum_{j>n} m^{(k)}(E_j)\rangle| \leq$$

$$\leq \sum_{j>n} [|\sigma_{M(E_j)}(x')| + |\sigma_{M(E_j)}(-x')|] < \varepsilon,$$

by taking the net limit we get $|\langle x', m^{(\infty)}(\bigcup_{n=1}^{\infty} E_n)\rangle - \langle x', \sum_{j=1}^{n} m^{(\infty)}(E_j)\rangle| \leq \varepsilon$. The arbitrariness of ε says that $\sum_{j=1}^{n} m^{(\infty)}(E_j) \xrightarrow{w} m^{(\infty)}(\bigcup_{n=1}^{\infty} E_n)$, this and the Orlicz-Pettis theorem imply that $\sum_{n=1}^{\infty} m^{(\infty)}(E_n)$ unconditionally converges to $m^{(\infty)}(\bigcup_{n=1}^{\infty} E_n)$. So we have shown that $m^{(\infty)}$ is an X-valued measure with $m^{(\infty)}(\Omega) = x$. □

Corollary 2.3. *Suppose that $M(\Sigma \to P_{wee}(X))$ is bounded and countably additive. Then the range of M, namely $M(\Sigma) \equiv \bigcup_{E\in\Sigma} M(E)$, is relatively weakly compact.*

Proof: Using an argument similar to Hiai [7, Corollary 2.4], it is immediately obtained by Theorem 2.2. □

Lemma 2.4. *Suppose that $\{E_n\}$ is a mutually disjoint sequence in Σ, suppose that $\{A_n\}$ is any sequence in Σ and suppose, further, that $m(\Sigma \to X)$ is strongly additive. Then for all $\varepsilon > 0$ there exists a positive integer k_0 such that $\|\sum_{k=k_1}^{k_2} m(A_n \cap E_k)\| < \varepsilon$ for $n = 1, 2, \ldots$, and for any integers k_1, k_2 with $k_0 \le k_1 \le k_2$.*

Proof: Suppose, to the contrary, that there is $\varepsilon_0 > 0$ such that the integer k_0 does not exist.

Inductively, by letting $k_0 = 1, 2, \ldots$, we obtain three positive integer sequences $\{n_j\}$, $\{k_{n_j}^{(1)}\}$ and $\{k_{n_j}^{(2)}\}$ with $k_{n_j}^{(1)} < k_{n_j}^{(2)} < k_{n_{j+1}}^{(1)}$ satisfying

$$\|\sum_{k_{n_j}^{(1)}}^{k_{n_j}^{(2)}} m(A_{n_j} \cap E_k)\| \ge \varepsilon_0. \tag{2.2}$$

The strong additive implies that the series $\sum_{j=1}^{\infty}(\sum_{k=k_{n_j}^{(1)}}^{k_{n_j}^{(2)}} m(A_{n_j} \cap E_k))$ unconditionally converges by noting that $\{\bigcup_{k=k_{n_j}^{(1)}}^{k_{n_j}^{(2)}} (A_{n_j} \cap E_k)\}$ is also a mutually disjoint sequence in Σ. This contradicts (2.2). □

Remark 2.5. Under the condition of Lemma 2.4, one can show that for any $\varepsilon > 0$ there is an integer $k_0 \ge 0$ such that

$$\|\sum_{k=k_0+1}^{\infty} m(A_n \cap E_k)\| < \varepsilon \quad \text{for} \quad n = 1, 2, \ldots .$$

Lemma 2.6. *Suppose that X has RNP and suppose that $m(\Sigma \to X)$ is σ-bounded variation X-valued measure. Then the range $m(\Sigma)$ of m is relatively compact.*

Proof: Suppose that $\{E_n\}$ is a Σ-partition of Ω satisfying $|m|(E_n) < \infty$ for $n = 1, 2, \ldots$, then m restricted to $\Sigma|_{E_n} \equiv \{E \cap E_n; E \in \Sigma\}$ is of bounded variation and $m(\Sigma|_{E_n})$ is relatively compact (see, for instance, Uhl [12]). We will show that $m(\Sigma)$ is relatively compact. It suffices to prove that $\{m(F_n)\}$ has convergent subsequences for any $\{F_n\} \subset \Sigma$. For every fixed integer $k \ge 1$, there is a subsequence $\{F_{n,k}\}$ of $\{F_n\}$ such that $m(F_{n,k} \cap E_k)$ converges. Since $m(F_n \cap E_k) \in m(\Sigma|_{E_k})$ and $m(\Sigma|_{E_k})$ is relatively compact, by a standard diagonal process one can claim a subsequence $\{F_{n,n}\} \subset \{F_n\}$ such that $\{m(F_{n,n} \cap E_k)\}$ converges for $k = 1, 2, \ldots$. Suppose $m(F_{n,n} \cap E_k) \to x_k$ for $k = 1, 2, \ldots$, due to Lemma 2.4, for any $\varepsilon > 0$ there is an integer $k_0 \ge 1$ such that $\| \sum_{k=k_1}^{k_2} m(F_{n,n} \cap E_k)\| < \varepsilon$ for $n = 1, 2, \dots$ whenever $k_0 \le k_1 \le k_2$. Hence $\| \sum_{k_1}^{k_2} x_k\| = \lim_{n\to\infty} \| \sum_{k_1}^{k_2} m(F_{n,n} \cap E_k)\| \le \varepsilon$, that is, $\sum_{k=1}^{\infty} x_k$ converges. Setting $x = \sum_{k=1}^{\infty} x_k$, as follows, we show $m(F_{n,n}) \to x$. Let integer $k_0 \ge 1$ satisfy (by Remark 2.5)

$$\| \sum_{k>k_0} m(F_{n,n} \cap E_k)\| < \varepsilon/3 \quad \text{and} \quad \| \sum_{k>k_0} x_k\| < \varepsilon/3 \quad \text{for} \quad n = 1, 2, \dots \tag{2.3}$$

and let integer $n_o \ge 1$ be such that

$$\|m(F_{n,n} \cap E_k) - x_k\| < \frac{\varepsilon}{3k_0} \quad \text{for} \quad k = 1, 2, \dots, k_0 \tag{2.4}$$

whenever $n \ge n_0$. Combining (2.3) and (2.4) together we obtain

$$\|m(F_{n,n}) - x\| \le \sum_{k=1}^{k_0} \|m(F_{n,n} \cap E_k) - x_k\| + \| \sum_{k>k_0} m(F_{n,n} \cap E_k)\| + \| \sum_{k>k_0}^{\infty} x_k\| < \varepsilon.$$

Therefore $m(F_{n,n}) \to x$. □

Theorem 2.7. *Suppose that X has RNP and suppose that $M(\Sigma \to P_a(X))$ is compact-valued, countably additive and of σ-bounded variation. Then the range $M(\Sigma)$ of M is relatively compact.*

Proof: Since M is compact-valued, the Mazur Theorem says that $\overline{\text{co}}M(E)$ is compact and convex for all $E \in \Sigma$. By Proposition 1.8, $\overline{\text{co}}M$ is countably additive and it is easy to observe it is of σ-bounded variation. It follows from Theorem 2.2 and the fact we just mentioned that there exists a σ-bounded variation selection m of $\overline{\text{co}}M$. therefore $m(\Sigma)$ is relatively compact by Lemma 2.6. Note that $\overline{\text{co}}M(E) + \overline{\text{co}}M(\Omega \setminus E) = \overline{\text{co}}M(\Omega)$. This implies that $M(E) \subset \overline{\text{co}}M(\Omega) - m(\Omega \setminus E) \subset \overline{\text{co}}M(\Omega) - m(\Sigma)$, and that $M(\Sigma)$ is relatively compact. □

3. Extension of set-valued measures

Suppose that $m_\tau(\mathcal{F} \to X, \tau \in T)$ are finitely additive, we say $\{m_\tau\}_{\tau\in T}$ are uniformly strongly additive provided $\sum_{n=1}^{\infty} m_\tau(E_n)$ converges unconditionally and uniformly for $\tau \in T$, for any mutually disjoint sequence $\{E_n\}$ in $\mathcal{F}$. Using an argument similar to the one for vector-valued measure ([5]), we have

Lemma 3.1. *Suppose that $M(\mathcal{F} \to P_{wee}(X))$ is finitely additive. Then the following versions are equivalent:*

(i) *M is strongly additive;*
(ii) *$\{\sigma_{M(\cdot)}(x'); \|x'\| \le 1, x' \in X'\}$ are uniformly strong additive;*
(iii) *for any mutually disjoint $\{E_n\}$ in $\mathcal{F}$, $\lim_{n\to\infty} |M(E_n)| = 0$.*

Lemma 3.2. *Suppose that $M(\mathcal{F} \to P_{wee}(X))$ is bounded and finitely (strongly) additive. Then for any $x \in M(A)$ and $A \in \mathcal{F}$ there is a bounded and finitely (strongly, respectively) additive measure $m(\mathcal{F} \to X)$ satisfying $m(A) = x$ and $m(E) \in M(E)$ for all $E \in \mathcal{F}$.*

PROOF: The proof is very much like of Theorem 2.2. □

Lemma 3.3. *Suppose that $M(\mathcal{F} \to P_{wee}(X))$ is bounded and finitely additive. Then M is strongly additive if and only if for any monotone non-decreasing sequence $\{E_n\}$ in $\mathcal{F}$ there is a relatively weakly compact set A in X such that $\lim_{n\to\infty} \sigma_{M(E_n)}(x') = \sigma_A(x')$.*

PROOF: Sufficiency. Suppose that $\{A_k\}$ in $\mathcal{F}$ is any mutually disjoint set sequence. Let $E_n = \bigcup_{k=1}^n A_k$, clearly, $\{E_n\}$ is monotone non-decreasing, by the hypotheses we obtain that there is a relatively weakly compact set A in X such that $\lim_{n\to\infty} \sigma_{M(E_n)}(x') = \sigma_A(x')$ for all $x' \in X'$. So one direction is shown by noting that $\lim \sigma_{M(E_n)}(x') = \lim_{n\to\infty} \sum_{k=1}^n \sigma_{M(A_k)}(x') = \sum_{n=1}^\infty \sigma_{M(A_n)}(x')$ and by Lemma 1.3.

Necessity. Suppose that $\{E_n\}$ in $\mathcal{F}$ is monotone non-decreasing. The strong additive of M implies that $\sigma_{M(E_n)}(x') = \sum_{k=0}^{n-1} \sigma_{M(E_{k+1}\setminus E_k)}(x')$ where $E_0 = \emptyset$ and that the series $\sum_{n=1}^\infty \sigma_{M(E_{k+1}\setminus E_k)}(x')$ converges for all $x' \in X'$. Therefore, $\lim_{n\to\infty} \sigma_{M(E_n)}(x') = \sum_{k=0}^\infty \sigma_{M(E_{k+1}\setminus E_k)}(x')$. Since $|\sum_{k=0}^n \sigma_{M(E_{k+1}\setminus E_k)}(x') - \sigma_{\sum_{k=0}^\infty M(E_{k+1}\setminus E_k)}(x')| \le |\sum_{k>n}^\infty \sigma_{M(E_{k+1}\setminus E_k)}(x')| + |\sum_{k>n}^\infty \sigma_{M(E_{k+1}\setminus E_k)}(-x')|$ for all positive integers n, we have $\sum_{k=0}^\infty \sigma_{M(E_{k+1}\setminus E_k)}(x') = \sigma_{\Sigma M(E_{k+1}\setminus E_k)}(x')$, taking $A = \sum_{k=0}^\infty M(E_{k+1} \setminus E_k)$, then $\lim_{n\to\infty} \sigma_{M(E_n)}(x') = \sigma_A(x')$. Since $M(E_{k+1} \setminus E_k)$ is weakly compact and $\sum_{k=0}^\infty M(E_{k+1} \setminus E_k)$ unconditionally convergent to A, A must be relatively weakly compact.

□

Lemma 3.4. $P_w(X) = \{A \subset P_{wee}(X)$; *$A$ is contained in a fixed weakly compact and convex set $W\}$ is complete corresponding to H (where H denotes the Hausdorff metric).*

PROOF: Suppose that $\{A_n\}$ in $P_w(X)$ is a Cauchy sequence, then there is a bounded set A in X such that $A_n \xrightarrow{H} A$ by noting that $\{B \subset X$; B is bounded, closed and convex$\}$ corresponding to H. Since $H(A, B) = \mathrm{SUP}_{\|x'\|\le 1} |\sigma_A(x') - \sigma_B(x')|$ (see, for instance, [4, Theorem II–18]), $\lim_{n\to\infty} \sigma_{A_n}(x') = \sigma_A(x')$ for all $x' \in X'$. The fact that $A_n \subset W$ implies $\sigma_A(x') \le \sigma_W(x')$ for all $x' \in X'$ which implies that $A \subset \overline{\mathrm{co}} W = W$. □

Theorem 3.5. *Suppose that Σ is a σ-algebra generated by $\mathcal{F}$ and suppose that $M(\mathcal{F} \to P_{wee}(X))$ is bounded and weakly countably additive. Then the following versions are equivalent:*

(i) *there is a unique extension $\overline{M}(\Sigma \to P_{wee}(X))$ of M which is countably additive;*

(ii) *there exists some non-negative real valued measure μ on Σ such that M is continuous to μ, that is,* $\lim_{\mu(E)\to 0} |M(E)| = 0$*;*
(iii) *M is strongly additive;*
(iv) *$M(\mathcal{F})$ is relatively weakly compact.*

Proof: (ii) $\Rightarrow$ (iii). By Lemma 3.1, it is easy to observe that this direction is true.

(iii) $\Rightarrow$ (iv). Lemma 3.2 implies that there is a strongly additive $m(\mathcal{F} \to X)$ such that $m(E) \in M(E)$ for all $E \in \mathcal{F}$, and $m(\mathcal{F})$ is relatively weakly compact by [5]. We obtain $M(\mathcal{F}) \subset M(\Omega) - m(\mathcal{F})$ by noting $M(E) + M(\Omega \setminus E) = M(\Omega)$ for any $E \in \mathcal{F}$, hence $M(\mathcal{F})$ is also relatively weakly compact.

(iv) $\Rightarrow$ (iii). Let $\{E_n\}$ be a monotone non-decreasing sequence in $\mathcal{F}$. By the boundedness and finite additivity, $\lim_{n\to\infty} \sigma_{M(E_n)}(x') = \sum_{n=0}^{\infty} \sigma_{M(E_{n+1}\setminus E_n)}(x')$ for all $x' \in X'$ where $E_0 = \emptyset$. Now we show that $\lim_n \sigma_{M(E_n)}(x') = \sigma_{\mathrm{ls}\, M(E_n)}(x')$ for all $x' \in X'$ as follows, where $\mathrm{ls}\, M(E_n) = \{x \in X;\ x = w - \lim_k x_{n_k}$, for some $x_{n_k} \in M(E_{n_k})$ and for all integers $k \geq 1\}$. Relatively weak compactness of $M(\mathcal{F})$ says there is a weakly compact set W such that $M(\mathcal{F}) \subset W$, in particular, $M(E_n) \subset W$ for $n = 1, 2, \dots$. For any $x' \in X'$, choose $x_n \in M(E_n)$ such that $\sigma_{M(E_n)}(x') = \langle x', x_n\rangle$ for $n = 1, 2, \dots$. Without loss of generality we can assume that $x_n \xrightarrow{w} x$ (otherwise we can choose a weakly convergent subsequence since W is weakly compact and $\{x_n\}$ in W), that is, $x \in \mathrm{ls}\, M(E_n)$. Thus

$$\lim_n \sigma_{M(E_n)}(x') = \langle x', x\rangle \leq \sigma_{\mathrm{ls}\, M(E_n)}(x').$$

On the other hand, for any $y \in \mathrm{ls}\, M(E_n)$, there is $\{y_{n_k} \in M(E_{n_k})\}$ such that $y_{n_k} \xrightarrow{w} y$, so we have

$$\langle x', y\rangle = \lim_n \langle x', y_{n_k}\rangle \leq \lim_n \sigma_{M(E_n)}(x').$$

That is, $\sigma_{\mathrm{ls}\, M(E_n)}(x') \leq \lim_n \sigma_{M(E_n)}(x')$, and further we have $\lim_n \sigma_{M(E_n)}(x') = \sigma_{\mathrm{ls}\, M(E_n)}(x')$. Since $M(E_n) \subset W$, it implies that $\mathrm{ls}\, M(E_n) \subset W$, and M is strongly additive by Lemma 3.3.

(iii) $\Rightarrow$ (ii). $\{\sigma_{M(\cdot)}(x') : x' \in X', \|x'\| \leq 1\}$ is uniformly strongly additive by Lemma 3.1. The Carathéodory-Hahn extension theorem implies that there is a unique countably additive extension $\overline{\sigma}_{M(\cdot)}(x')$ of $\sigma_{M(\cdot)}$ on Σ. According to [5, Lemma 1, p. 26 and Theorem 4, p. 11] there exists a non-negative real valued measure μ on Σ such that $\lim_{\mu(E)\to 0} \sigma_{M(E)}(x') = 0$ uniformly on $B = \{x' \in X'; \|x'\| \leq 1\}$, that is, $\lim_{\mu(E)\to 0} |M(E)| = 0$.

(ii) & (iv) $\Rightarrow$ (i). Let μ be a non-negative real valued measure satisfying (ii), and let W be a weakly compact and convex set in X such that $M(\mathcal{F}) \subset W$. We define the pseudo-metric ρ on Σ by $\rho(E_1, E_2) = \mu(E_1 \vartriangle E_2)$ for $E_1, E_2 \in \Sigma$ where

$\triangle$ denotes the symmetric difference operation. We denote by $\Sigma(\mu)$ the pseudo-metric space equipped with ρ on Σ. Since Σ is generated by $\mathcal{F}$, the pseudo-metric space $\mathcal{F}(\mu)$, the restriction of ρ to $\mathcal{F}$, is a dense subspace of $\Sigma(\mu)$. Next, we define the mapping M: $\mathcal{F}(\mu) \to P_w$ (the family of nonempty weakly compact sets which are contained in a weakly compact and convex set W equipped with the Hausdorff metric H) by $E \to M(E)$, we will show that M is uniformly continuous on $\mathcal{F}(\mu)$. First, we prove the following inequality

$$\begin{aligned} H(M(E_1), M(E_2)) &\le H(M(E_2 \setminus (E_1 \cap E_2)), \{0\}) \\ &\quad + H(M(E_1 \setminus (E_1 \cap E_2)), \{0\}) \\ &\equiv |M(E_2 \setminus (E_1 \cap E_2))| + |M(E_1 \setminus (E_1 \cap E_2))|. \end{aligned} \tag{3.1}$$

For $x_j \in M(E_j)$ $(j = 1, 2)$, by Lemma 3.2 there exists a finitely additive set function m such that $m(E_2) = x_2$ and such that $m(E) \in M(E)$ for all $E \in \mathcal{F}$. Due to the equation $m(E_1) + m(E_2 \setminus (E_1 \cap E_2)) = m(E_2) + m(E_1 \setminus (E_1 \cap E_2))$, we have

$$\begin{aligned} d(x_2, M(E_1)) &= \inf_{x \in M(E_1)} \|x_2 - x\| \le \|x_2 - m(E_1)\| \\ &\le \|m(E_2 \setminus (E_1 \cap E_2))\| + \|m(E_1 \setminus (E_1 \cap E_2))\| \\ &\le |M(E_2 \setminus (E_1 \cap E_2))| + |M(E_1 \setminus (E_1 \cap E_2))| \end{aligned}$$

and similarly we have

$$d(x_1, M(E_2)) \le |M(E_2 \setminus (E_1 \cap E_2))| + |M(E_1 \setminus (E_1 \cap E_2))|.$$

Combining the two inequalities together we proved that (3.1) holds. Both (ii) and inequality (3.1) imply that M is uniformly continuous.

Note that $(P_w(X), H)$ is a complete metric space (Lemma 3.4), hence there is a uniformly continuous extension $\overline{M}[\Sigma(\mu) \to (P_w(X), H)]$ of M from $\mathcal{F}(\mu)$ to $\Sigma(\mu)$. Let $\{E_n\}$ be any mutually disjoint sequence in Σ, then $H(\overline{M}(\bigcup_{k=1}^{\infty} E_k),\ \sum_{k=1}^{n} \overline{M}(E_k)) \to 0$. Since $H(A, B) = \mathrm{SUP}_{\|x'\| \le 1} |\sigma_A(x') - \sigma_B(x')|$, we have $\sigma_{\overline{M}(\bigcup_{n=1}^{\infty} E_n)}(x') = \sum_{n=1}^{\infty} \sigma_{\overline{M}(E_n)}(x')$. That is, $\overline{M}$ is weakly countably additive, it follows from Corollary 1.5 that $\overline{M}$ is countably additive.

$\square$

4. Integral representation and set valued measures

In this section, X will always be a separable Banach space, (Ω, Σ, μ) denotes a complete and finite measure space and $F(\Omega \to P_f(X)$, the family of all nonempty and closed sets in $X)$ denotes a set valued function. The graph of F is denoted by $G_R F = \{(\omega, x) \in \Omega \times X; x \in F(\omega)\}$. For $A \subset X$, we write $F^{-1}(A) = \{\omega \in \Omega; F(\omega) \cap A \neq \emptyset\}$. By [6], the following versions are equivalent:

(i) F is measurable;

(ii) for any $A \in P_f(X)$, $F^{-1}(A) \in \Sigma$;
(iii) there exists a sequence $\{f_n\}$ of measurable functions from Ω to X such that $F(\omega) = \mathrm{cl}\{f_n(\omega)\}$;
(iv) $G_R F \in \Sigma \times \mathcal{B}(X)$, where $\mathcal{B}(X)$ is the Borel σ-algebra on X.

We call a measurable function $\sigma(\Omega \to X)$ a measurable selection of F provided $\sigma(\omega) \in F(\omega)$ μ a.e.; such a σ is said to be a weakly integrable selection if it is Pettis-integrable. We set

$$S_{WF} = \{\sigma : \sigma \ \text{ is a weakly integrable selection of } F\}.$$

For $A \in \Sigma$, the Pettis-Aumann type integral of F is defined by $(W)\int_A F\,d\mu = \{\text{Pettis-}\int_A \sigma(\omega)\,d\mu; \sigma \in S_{WF}\}$. F is said to be weakly integrable bounded provided for each $x' \in X'$, $|x'F(\omega)| \equiv \sup_{x\in F(\omega)} |\langle x', x\rangle| \equiv f(\omega) \in L'(\mu)$.

All theorems and terminology about topological linear spaces of this section are referred to [13].

Lemma 4.1. *Suppose that $\sigma(X' \to R)$ is a sublinear functional (Minkowski gauge) which is continuous relative to the Makey topology $\tau(X', X)$. Then there is $A \in P_{wee}(X)$ such that $\sigma_A = \sigma$ on X' and $A = \{x \in X; \langle x', x\rangle \leq \sigma(x')$ for all $x' \in X'\}$.*

PROOF: Since $\sigma(x')$ is continuous about $\tau(X', X)$, it must be continuous by the norm topology. Therefore there is a closed and convex set A'' such that $A'' = \{x'' \in X''; \langle x'', x'\rangle \leq \sigma(x')$ for all $x' \in X'\}$. First, we show $A'' = A$. Clearly, $A \subset A''$. On the other hand, for any $x'' \in A''$, we have $\langle x'', x'\rangle \leq \sigma(x')$. That is, x'' is a $\tau(X', X)$ continuous linear functional on X', hence $x'' \in X$ and further we have $x'' \in A$. Thus $A'' = A$.

It remains to show that A is weakly compact. Clearly, A is bounded, convex and closed, therefore it is also weakly closed. Suppose, to the contrary, that A is not weakly compact, then, by James' theorem, there exist $x'_0 \in X'$ such that $\langle x'_0, x\rangle < \sigma_A(x'_0)$ for all $x \in A$. Let $\{x_\alpha\}$ be a net in A such that $\langle x'_0, x_\alpha\rangle \to \sigma_A(x')$, then there is a subnet $\{x_\beta\} \subset \{x_\alpha\}$ such that $x_\beta \xrightarrow{w^*} x''$ for some $x'' \in X''$, since $\{x_\alpha\}$ is bounded. It is easy to observe that $\langle x'', x'_0\rangle = \sigma_A(x'_0)$ and for all $x' \in X'$, $\langle x'', x'\rangle \leq \sigma_A(x')$, that is, $x'' \in A'' = A$. This contradicts our hypotheses. □

Lemma 4.2. *Suppose that $F(\Omega \to P_f(X))$ is measurable and suppose $S_{WF} \neq \emptyset$. Then $\sigma_{(W)\int_A F\,d\mu}(x') = \int_A \sigma_{F(\omega)}(x')\,d\mu$ for all $A \in \Sigma$ and $x' \in X'$.*

PROOF: Without loss of generality we assume $A = \Omega$. The measurability of F implies that $\sigma_{F(\omega)}(x')$ is also measurable and $\sigma_{(W)\int_\Omega F\,d\mu}(x') \leq \int_\Omega \sigma_{F(\omega)}(x')\,d\mu$. For each integer $n \geq 1$, let $E_n = \{\omega \in \Omega; \sigma_{F(\omega)}(x') \leq n\}$ and define a measurable function $f_n(\Omega \to R)$ by

$$f_n(\omega) = \begin{cases} \sigma_{F(\omega)}(x') - \frac{1}{n}, & \text{for } \omega \in E_n, \\ n, & \text{otherwise.} \end{cases}$$

Next, define $H_n(\omega)$ $(\Omega \to P_f(X))$ by

$$H_n(\omega) = \{x \in F(\omega); \langle x', x\rangle \geq f_n(\omega)\}.$$

Since $\theta(\omega, x) = \langle x', x\rangle - f_n(\omega)$ is continuous to x and measurable to ω, θ is a Carathéodory function, it must be $\Sigma \times \mathcal{B}(X)$-measurable, that is, $G_R H_n = G_R F \cap \{(\omega, x) \in \Omega \times X; \theta(\omega, x) \geq 0\} \in \Sigma \times \mathcal{B}(X)$. Hence $H_n(\omega)$ is measurable and there exists a measurable selection σ_n of H_n for $n = 1, 2, \ldots$. Define again $\sigma_{n,k}$ by

$$\sigma_{n,k}(\omega) = \begin{cases} \sigma_n(\omega), \omega \in \Omega_{n,k} \equiv \{\omega \in \Omega; \|\sigma_n(\omega)\| \leq k\}, \\ \sigma(\omega), \quad \text{otherwise} \end{cases}$$

where $\sigma \in S_{WF}$, hence $\sigma_{n,k} \in S_{WF}$. Since

$$\int_\Omega x'\sigma_{n,k}(\omega)\, d\mu = \int_{\Omega_{n,k}} x'\sigma_n(\omega)\, d\mu + \int_{\Omega\setminus\Omega_{n,k}} x'\sigma(\omega)\, d\mu,$$

we have

$$\sigma_{(W)\int_\Omega F\, d\mu}(x') \geq \int_{\Omega_{n,k}} x'\sigma_n(\omega)\, d\mu + \int_{\Omega\setminus\Omega_{n,k}} x'\sigma(\omega)\, d\mu.$$

Since $\mu(\Omega \setminus \Omega_{n,k}) \to 0$ as $k \to \infty$ and since $\sigma(\omega)$ is Pettis-integrable, by letting k tend to positive infinity in the above inequality we obtain

$$\begin{aligned}\sigma_{(W)\int_\Omega F\, d\mu}(x') &\geq \int_\Omega f_n(\omega)\, d\mu = \int_{E_n} (\sigma_{F(\omega)}(x') - \frac{1}{n})\, d\mu + n\mu(\Omega \setminus E_n) \\ &\geq \int_{E_n} (\sigma_{F(\omega)}(x') - \frac{1}{n})\, d\mu.\end{aligned}$$

Also, letting n go to infinity we have

$$\sigma_{(W)\int_\Omega F\, d\mu}(x') \geq \int_\Omega \sigma_{F(\omega)}(x')\, d\mu$$

which completes the proof. □

Definition 4.3. A bounded set valued measure $M(\Sigma \to P_{wee}(X))$ is said to be μ-weakly compactly separable, provided there exists a Σ-countable partition $\{\Omega_n\}$ of Ω such that $K_n = \{\frac{x}{\mu(A)}; x \in M(A), \mu(A) > 0, A \subset \Omega_n\}$ is relatively weakly compact for $n = 1, 2, \ldots$.

Theorem 4.4. *Suppose that X' is separable and suppose $M(\Sigma \to P_{wee}(X))$ is a set valued measure of μ-continuity. Then there exists a measurable and weakly integrable bounded set valued function $F(\Sigma \to P_{wee}(X))$ such that*

$$M(A) = (W)\int_A F\, d\mu$$

if and only if M is μ-weakly compactly separable.

Proof: Necessity. Set $\Omega_n = \{\omega \in \Omega; n-1 \leq |F(\omega)| < n\}$. The measurability of F implies that $\Omega_n \in \Sigma$ and that $\bigcup_{n=1}^{\infty} \Omega_n = \Omega$, that is, $\{\Omega_n\}$ is a Σ-countable partition of Ω. Let $K_n = \{\frac{x}{\mu(A)}; x \in M(A), \mu(A) > 0, A \subset \Omega_n \text{ and } A \in \Sigma\}$ and for any fixed $x' \in X'$ let $R_{x'}(\omega) = \{x \in F(\omega), \sigma_{F(\omega)}(x') = \langle x', x\rangle\}$, then $R_{x'}(\omega) \neq \emptyset$ for all $\omega \in \Omega$. Therefore there exists a measurable selection σ of $R_{x'}$. Since $\|\sigma(\omega)\| \leq |F(\omega)| < n$ on Ω_n, $\sigma(\omega)$ is Bochner-integrable on Ω_n for $n = 1, 2, \dots$. Choose any $\sigma_0 \in S_{WF}$ and define σ_1 by $\sigma_1(\omega) = \sigma(\omega)$, if $\omega \in \Omega_n$, $= \sigma_0(\omega)$, otherwise, hence $\sigma_1 \in S_{WF}$. For any $A \subset \Omega_n$, according to the fact we have just proved and Lemma 4.2, we have

$$\sigma_{M(A)}(x') = \int_A \sigma_{F(\omega)}(x')\, d\mu = \int_A \langle x', \sigma(\omega)\rangle\, d\mu = \langle x', \int_A \sigma_1(\omega)\, d\mu\rangle. \tag{4.1}$$

Without loss of generality we can assume that $S_n \equiv \{\frac{m(A)}{\mu(A)}; \mu(A) > 0, A \in \Sigma, A \subset \Omega_n\}$ is relatively weakly compact, since σ_1 is Bochner-integrable on Ω_n, where $m(A) = \int_A \sigma_1(\omega)\, d\mu$. The Krein-Smulian theorem implies that $\overline{\mathrm{co}}(S_n)$ is weakly compact and convex. Thus, by (4.1), $\sigma_{K_n}(x') = \sigma_{S_n}(x') = \sigma_{\overline{\mathrm{co}}(S_n)}(x')$, and there exists $x_n \in \overline{\mathrm{co}}(S_n) \subset \overline{\mathrm{co}}(K_n)$ such that $\sigma_{K_n}(x') = \langle x', x_n\rangle$ for $n = 1, 2, \dots$. This and the James' theorem say $\overline{\mathrm{co}}(K_n)$ is weakly compact. Because $F(\omega)$ is weakly integrable bounded, $|\sigma_{M(A)}(x')| \geq \int_\Omega |x'F(\omega)|\, d\mu$ for all $A \in \Sigma$, and M is bounded by the Resonance Theorem.

Sufficiency. Suppose that M is μ-weakly compactly separable. Let $\{\Omega_n\}$ be a Σ-countable partition on Ω such that $K_n = \{\frac{x}{\mu(A)}; x \in M(A), A \in \Sigma, \mu(A) > 0$ and $A \subset \Omega_n\}$ is relatively weakly compact, then $M(\Sigma_n \equiv \Sigma\,|_{\Omega_n} \to P_{wee}(X))$ is of bounded variation and μ continuous and which implies $\sigma_{M(\cdot)}(x')$ is also of bounded variation and μ-continuous on Σ_n for all $x' \in X'$. Since R has RNP, for each fixed integer $n \geq 1$ there exists $\varphi_n(x', \omega) \in L^1(\Omega_n)$ such that

$$\sigma_{M(A)}(x') = \int_A \varphi_n(x', \omega)\, d\mu. \tag{4.2}$$

Note that $|\sigma_{M(A)}(x')| \leq C_n\mu(A)$, where $C_n = \sup_{x\in K_n} \|x\|$, we know that the variation $|\sigma_{M(A)}|(x')$ of $\sigma_{M(\cdot)}(x')$ on A satisfies $|\sigma_{M(A)}|(x') \leq C_n\mu(A)\|x'\|$ for $A \subset \Omega_n$ and $A \subset \Sigma$. By (4.2), we have $|\sigma_{M(A)}|(x') = \int_A |\varphi_n(x', \omega)|\, d\mu$, so $|\varphi_n(x', \omega)| \leq C_n\|x'\|$ μ a.e. on Ω_n, that is, $\varphi_n(x', \omega) \in L^\infty(\Omega_n)$. By [8], there is a positive and linear lifting L on $L^\infty(\Omega_n)$ such that for each $f \in L^\infty(\Omega_n)$, $\overline{f}(\omega) \equiv L(f(\omega))$ is bounded and measurable function satisfying

$$\int_A \overline{f}(\omega)\, d\mu = \int_A f(\omega)\, d\mu \quad \text{and} \quad \sup_{\omega\in\Omega_n} \|\overline{f}(\omega)\| \leq \|f\|_\infty. \tag{4.3}$$

Write $\overline{\varphi}_n(x',\omega) = L(\varphi_n(x',\omega))$, then (4.3) and

$$\int_A \varphi_n(x'_1 + x'_2, \omega)\, d\mu \le \int_A \varphi_n(x'_1, \omega)\, d\mu + \int_A \varphi_n(x'_2, \omega)\, d\mu$$

together with

$$\int_A \varphi_n(\alpha x', \omega)\, d\mu = \alpha \int_A \varphi_n(x', \omega)\, d\mu \qquad (\text{for } \alpha \ge 0)$$

imply that $\overline{\varphi}_n(x',\omega)$ is a sublinear functional on X'. Let $W_n = (K_n \cup (-K_n))$, then W_n is absolutely convex and weakly compact. We obtain $|\sigma_{M(A)}|(x') \le \sigma_{W_n}(x')\mu(A)$ for all $A \subset \Sigma$, $M(A) \subset \mu(A)W_n$ by noting $|\sigma_{M(A)}(x')| \le \sigma_{W_n}(x')\mu(A)$. It follows from the Makey-Arens theorem that for fixed ω, $\overline{\varphi}_n(x',\omega)$ is continuous on X' corresponding to the Makey topology, by Lemma 4.1, for every $\omega \in \Omega_n$ there exists $F_n(\omega) \in P_{wee}(X)$ such that $F_n(\omega) = \{x \in X; \langle x', x\rangle \le \overline{\varphi}_n(x',\omega) \text{ for } x' \in X'\}$ and

$$\sigma_{F_n(\omega)}(x') = \overline{\varphi}_n(x',\omega). \tag{4.4}$$

Now define F on Ω by $F(\omega) = F_n(\omega)$ for $\omega \in \Omega_n$, equation (4.4) implies that $\sigma_{F_n(\omega)}(x')$ is measurable, this and [4] imply that $F_n(\omega)$ is measurable. This implies that $F(\omega)$ is measurable. Hence there is a measurable selection $\sigma(\Omega \to X)$ of F which is Bochner-integrable by noting $\|\sigma(\omega)\| = \sup_{\|x'\|\le 1} |\langle x', \sigma(\omega)\rangle| \le C_n$ on Ω_n. By (4.2) we have

$$\langle x', \int_A \sigma(\omega)\, d\mu\rangle \le \sigma_{M(A)}(x').$$

This and the Separation Theorem say $\int_A \sigma(\omega)\, d\mu \in M(A)$. Since $\int_{E\cap\Omega_n} \sigma(\omega)\, d\mu \in M(E\cap\Omega_n)$ for any $E \in \Sigma$, $\sum_{n=1}^{\infty} M(E\cap\Omega_n)$ unconditionally converges by Corollary 1.3, in particular, $\sum_{n=1}^{\infty} \int_{E\cap\Omega_n} \sigma(\omega)\, d\mu$ is unconditionally convergent. $\sigma(\omega)$ is Pettis-integrable by noting

$$\langle x', \sum_{n=1}^{\infty} \int_{E\cap\Omega_n} \sigma(\omega)\, d\mu\rangle = \sum_{n=1}^{\infty} \int_{E\cap\Omega_n} x'\sigma(\omega)\, d\mu = \int_E x'\sigma(\omega)\, d\mu,$$

that is, $\sigma \in S_{WF}$. Lemma 4.2 implies that

$$\sigma_{(W)\int_A F\, d\mu}(x') = \int_A \sigma_{F(\omega)}(x')\, d\mu \tag{4.5}$$

and on the other hand, combining (4.2)–(4.4) together we have

$$\int_A \sigma_{F(\omega)}(x')\, d\mu = \sigma_{M(A)}(x') \tag{4.6}$$

and (4.5), (4.6) and the Separation Theorem imply

$$M(A) = \mathrm{cl}\,\big((W)\int_A F\,d\mu\big). \tag{4.7}$$

The weak integrability of F can be followed by

$$\int |x'F(\omega)|\,d\mu \le \|x'\| \sup_{\|x'\|\le 1} |\sigma_{M(A)}|(x') < \infty.$$

It remains to show that $(W)\int_A F\,d\mu$ is closed. Suppose that $\{\sigma_n(\omega)\} \subset S_{WF}$ such that $x_0 = \lim_{n\to\infty}\int_A \sigma_n(\omega)\,d\mu$. Let $\{x'_n\}$ be a countably dense set in X' (X' is separable). The inequality $|x'_1\sigma(\omega)| \le |x'_1F(\omega)|$ and the Dunford theorem (see, for instance, [5, Theorem 15]) imply that $\{x'_1\sigma_n(\omega)\}$ is relatively weakly compact set in $L'(\mu)$. So there exists a weakly convergent subsequence (we still denote it by $\{x'_1\sigma_n(\omega)\}$). By the Mazur theorem, there exists a function sequence $\{f_{1,k}(\omega)\}$ satisfying $f_{1,k}(\omega) \in \mathrm{co}\{\sigma_k(\omega), \sigma_{k+1}(\omega), \dots\}$ such that $x'(f_{1,k}(\omega))$ is norm-convergent in $L'(\mu)$, this implies that there exists $E_1 \in \Sigma$ with $\mu(E_1) = 0$ such that $x'_1(f_{1,k}(\omega))$ is pointwise convergent on $\Omega \setminus E_1$. The convexity of F implies that $f_{1,k}(\omega) \in F(\omega)$ and clearly, it also satisfies $|x'_2 f_{1,k}(\omega)| \le |x'_2F(\omega)|$ which implies that there exists a function sequence $\{f_{2,k}(\omega)\}$ and $E_2 \in \Sigma$ with $\mu(E_2) = 0$ such that $f_{2,k}(\omega) \in \mathrm{co}\{f_{1,k}(\omega), f_{1,k+1}(\omega), \dots\}$ and such that $x'_2 f_{2,k}(\omega)$ pointwise converges on $\Omega \setminus E_2 \dots$. Inductively, we obtain a function sequence $\{f_{n,k}(\omega)\}$ and set sequence $\{E_n\} \subset \Sigma$ with $\mu(E_n) = 0$ for $n = 1, 2, \dots$, such that

(a) $f_{n+1,k}(\omega) \in \mathrm{co}\{f_{n,k}(\omega), f_{n,k+1}(\omega), \dots\}$;
(b) $x'_n f_{n,k}$ is pointwise convergent on $\Omega \setminus E_n$.

Let $g_k(\omega) = f_{k,k}(\omega)$, and let $E_0 = \bigcup_{n=1}^{\infty} E_n$, therefore for $\omega \in \Omega \setminus E_0$, $\lim_{k\to\infty} x'_n g_k(\omega)$ exists (for $n = 1, 2, \dots$) by combining (a) and (b). For any $x' \in X'$, $|x'_n g_{k_1}(\omega) - x' g_{k_2}(\omega)| \le \|x'_n - x'\|\,\|g_{k_1}(\omega)\| + |x'_n g_{k_1}(\omega) - x'_n g_{k_2}(\omega)|$ for any integers $k_1, k_2 \ge 1$, the density of $\{x'_n\}$ in X' implies that $\lim_{k\to\infty} x' g_k(\omega)$ exist for all $x' \in X'$ and $\omega \in \Omega \setminus E_0$. Let $K(\omega) = \overline{\mathrm{co}}(F(\omega) \cup (-F(\omega)))$, then $K(\omega)$ is absolutely convex and weakly compact for every $\omega \in \Omega \setminus E_0$. Since $|\lim_{k\to\infty} x' g_k(\omega)| \le \sigma_{K(\omega)}(x')$ for fixed ω, $\lim_{k\to\infty} x' g_k(\omega)$ is continuous corresponding to the Makey topology on X'. Thus, there exists a function $g : (\Omega \setminus E_0 \to X)$ satisfying $\langle x', g(\omega)\rangle = \lim_{k\to\infty} x' g_k(\omega)$. Again by the Mazur theorem we know $g(\omega) \in F(\omega)$. Choose any $\sigma \in S_{WF}$ and define σ_1 by $\sigma_1(\omega) = g(\omega)$, $\omega \in \Omega \setminus E_0; = \sigma(\omega)$, otherwise; let $\{B_k\}$ be a Σ-countable partition and let σ_1 be Bochner-integrable on B_k, then, by (4.7), $\int_{B_k} \sigma_1(\omega)\,d\mu \in M(B_k)$. So $\sum_{k=1}^{\infty}\int_{B_k\cap E} \sigma(\omega)$ is Pettis-integrable, that is $\sigma_1(\omega) \in S_{WF}$. On the other hand, $\langle x', x_0\rangle = \lim_{n\to\infty}\int_A x'\sigma_n(\omega)\,d\mu = \int_A x' g(\omega)\,d\mu = \int_A x'\sigma_1(\omega)\,d\mu$, therefore $x_0 = \int_A \sigma_1(\omega)\,d\mu \in (W)\int_A F\,d\mu$, which completes our proof.

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