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Relatively compact spaces and separation properties

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Abstract. We consider the property of relative compactness of subspaces of Hausdorff spaces. Several examples of relatively compact spaces are given. We prove that the property of being a relatively compact subspace of a Hausdorff spaces is strictly stronger than being a regular space and strictly weaker than being a Tychonoff space.

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Notations and terminology follow [En] and [PW]. All spaces under consideration are assumed to be Hausdorff topological spaces. Condensation is a one-to-one continuous map onto. Cardinals are initial ordinals. Symbols ω, $\mathbb{Z}$, $\mathbb{R}$ stand for the first infinite cardinal, the set of all integers and the real line, respectively. A space, closed in every regular (Hausdorff) space containing it is called *R-closed* (*H-closed*). A subspace Y of a space X is said to be *relatively compact* in X iff every open cover of X has a finite subcover of Y [Ra]. A subspace Y of a space X is said to be *relatively normal* in X iff whenever F_1 and F_2 are closed subsets of Y and $\mathrm{Cl}_X F_1 \cap \mathrm{Cl}_X F_2 = \varnothing$, then there are disjoint open subsets U_1 and U_2 of X, such that $F_1 \subset U_1$ and $F_2 \subset U_2$.

Every relatively compact subspace of a space X is relatively normal in X and every relatively normal subspace is a regular space [AH]. On the other hand, every subspace Y of a compact space K is relatively compact in K. Hence every Tychonoff space Y can be embedded into some space (e.g. $I^{w(Y)}$) as a relatively compact subspace. Therefore we could consider *"being a relatively compact subspace"* as a separation property, between regularity and complete regularity. Below we shall show that our property is *strictly* stronger than regularity and *strictly* weaker than complete regularity. We also use the following separation property.

1.1 Definition. A space X has *the countable separation property* iff whenever F is a closed subspace of X and $x \notin F$, there are open $W_i : i \in \omega$ such that for each $i \in \omega$ $x \notin W_i$, $F \subset W_i$ and $\mathrm{Cl}_X W_{i+1} \subset W_i$.

Clearly, every Tychonoff space has the countable separation property and each space with countable separation property is regular.

1.2 Definition. A space Y will be said to be *potentially compact*, if there is a space X such that Y is a subspace of X and Y is relatively compact in X.

Thus we have

1.3 Proposition [AH]**.** *Every potentially compact space is regular.*

The following observation helps to identify several regular spaces which are not relatively compact in any Hausdorff space.

1.4 Proposition. *Let Y be an R-closed space which is relatively compact in a space X. Then Y is compact.*

Proof: Choose arbitrary $x \in X \setminus Y$ and let $Y_1 = Y \cup \{x\}$. Clearly, Y_1 is relatively compact in X. Hence Y_1 is regular (Proposition 1.3). Then Y is closed in Y_1. So, $x \notin \mathrm{Cl}_X\, Y$. It follows that Y is closed in X. Thus Y is compact in itself, i.e. Y is compact. $\square$

So, any regular R-closed non-compact space is not relatively compact in any Hausdorff space containing it. One of the well-known examples with such properties is the Jones space over $Y = (\omega_1 + 1) \times (\omega_1 + 1) \setminus \{(\omega_1, \omega_1)\}$ [Jo], see also [PW, p. 150–153]. Let $C = \omega_1 \times \{0\}$, $D = \{0\} \times \omega_1$, $\bar{Y}$ be the quotient space obtained from $Y \times \omega$ by identifying C_{2n+1} with C_{2n+2} and D_{2n+2} with D_{2n+3} for each $n \in \mathbb{N}$ and $q : Y \times \omega \to \bar{Y}$ be the natural quotient map. Let $\tilde{Y} = \bar{Y} \cup \{z\}$ topologized as follows: $\bar{Y}$ is an open subspace of $\tilde{Y}$, and $\big\{\{z\} \cup \cup_{n>k} Y_n :\ k \in \omega\big\}$ is a base in z. The resulting space is regular, not completely regular space [Jo], see also [PW, p. 150–153].

1.5 Proposition. *Let $\tilde{Y}$ be the Jones space over $(\omega_1+1)\times(\omega_1+1)\setminus\{(\omega_1,\omega_1)\}$. Then $\tilde{Y}$ is not relatively compact in any Hausdorff space.*

Proof: In view of Proposition 1.4 we need to prove only that $\tilde{Y}$ is R-closed. Assume the contrary: X is a regular space, $\tilde{Y} \subset X$ and $x \in \mathrm{Cl}_X\, \tilde{Y} \setminus \tilde{Y}$. Clearly $x \in \mathrm{Cl}_X\, Y_n$ for some $n \in \omega$. Now we need the following fact:

1.6 Claim. *Let X be a regular space and $Y = (\omega_1+1)\times(\omega_1+1)\setminus\{(\omega_1,\omega_1)\} \subset X$. Then $|X\setminus Y| \le 1$. If, moreover, $X\setminus Y \ne \varnothing$, then $\mathrm{Cl}_X\, Y = (\omega_1+1)\times(\omega_1+1) = \beta Y$.*

It follows that $x \in \mathrm{Cl}_X\, Y_{n+1}$, and by induction we have that $x \in \mathrm{Cl}_X\, Y_k$ for each $k \ge n$. So, $x = z$ contradicting $x \notin \tilde{Y}$. $\square$

1.7 Remark. The above arguments also work to show that $\tilde{Y}$ is not relatively normal in any regular space.

To construct a non-Tychonoff space Y which is relatively compact in some Hausdorff space X, we need the following lemma.

1.8 Lemma. *There are a Hausdorff space X and a Tychonoff zero-dimensional relatively compact subspace Y of X and two uncountable closed disjoint G_δ-subsets F_1 and F_2 of a space Y, such that $\mathrm{Cl}_X\, F_1 \cap \mathrm{Cl}_X\, F_2 = \varnothing$, F_1 and F_2 can be separated (in Y) by disjoint open sets, but whenever $f : Y \to \mathbb{R}$ is a continuous function then $|f^{-1}(0) \cap F_1| > \omega$ implies $|F_2 \setminus f^{-1}(0)| \le \omega$. In particular, F_1 and F_2 cannot be separated (in Y) by a continuous real-valued function.*

Proof: Let Y be the set

$$[-1,1] \times [0,1] \setminus \{(-1,0),(1,0)\}.$$

Basic elements for topology of Y are either:
1. $\{x\}$ for $x \in (-1,1) \times (0,1]$;
2.
$$\{[-1,-1+\varepsilon) \times \{y\} : 0 < \varepsilon < 1\}$$
for $(x,y) \in \{-1\} \times (0,1]$;
$$\{(1-\varepsilon,1] \times \{y\} : 0 < \varepsilon < 1\}$$
for $(x,y) \in \{1\} \times (0,1]$;
3.
$$\big\{\{(x+e(1-|x|)t,t) : t \in [0,1] \setminus K,\ e \in \{-1/2,0,1/2\}\}\, K \in [(0,1]]^{<\omega}\big\}$$
for $(x,y) \in (-1,1) \times \{0\}$.

A typical neighborhood V_a of a point $(a,0)$ can be described in the following way. Take the vertical line $l_0 :\ x = a$ through $(a,0)$ and the two lines l_+ and l_- through $(a,0)$ symmetrical with respect to l_0 having the slope $\pm 2/(1-|a|)$. Then V_a is the intersection of the union $l_0 \cup l_+ \cup l_-$ with the rectangle $[-1,1] \times [0,1]$ from which any finite set of points different from $(a,0)$ is removed.

Clearly Y is a Hausdorff zero-dimensional (hence Tychonoff) space. Let
$$\begin{aligned} F_1 &= \{-1\} \times (0,1] \\ F_2 &= \{1\} \times (0,1] \\ U_1 &= [-1,-1+1/10) \times (0,1] \\ U_2 &= (1-1/10,1] \times (0,1]. \end{aligned}$$
Then F_1, F_2 are disjoint closed G_δ-subsets of Y, U_1, U_2 are disjoint open neighborhoods of F_1 and F_2 respectively. Moreover for
$$W_i = \left[-1,-1+\frac{1}{2^{2i+1}}\right) \times (0,1] \ :\ \ i \in \omega$$
we have: $\cap_{i\in\omega} W_i = F_1$ and $\mathrm{Cl}_X\, W_{i+1} \subset W_i$.

First we prove

1.9 Claim. *Let $f : Y \to \mathbb{R}$ be a continuous function such that $|f^{-1}(0) \cap F_1| > \omega$. Then $|F_2 \setminus f^{-1}(0)| \le \omega$.*

Proof: Assume the contrary: $f : Y \to \mathbb{R}$ is a continuous function such that $|f^{-1}(0) \cap F_1| > \omega$ and $|F_2 \setminus f^{-1}(0)| > \omega$. Then there are $\varepsilon > 0$ and $P \in [F_2]^{>\omega}$ such that $\forall p \in P\ f(p) > 3\varepsilon$. Since f is continuous there are $\delta > 0$, $L \in [P]^{>\omega}$, $M \in [F_1]^{>\omega}$ such that $f(x,y) > 2\varepsilon$ whenever $1-\delta < x < 1$, $y \in L$ and $f(x,y) < \varepsilon$ whenever $-1 < x < -1+\delta$, $y \in M$. By the definition of the base of Y and continuity of f we have $f(x,0) < \varepsilon$ for each $-1 < x < -1+\delta$ and $f(x,0) > 2\varepsilon$

for each $1-\delta < x < 1$. Moreover, there is a family $\{K_x : x \in (-1,-1+\delta)\} \subset \big[(0,1]\big]^{<\omega}$ such that

$$f\left(\bigcup\Big\{\{(x+e(1-|x|)t,t) : t \in [0,1] \setminus K_x,\ e \in \{\tfrac{-1}{2},0,\tfrac{1}{2}\}\}x \in (-1,-1+\delta)\Big\}\right) \subset [0,\varepsilon).$$

It follows that $f(x,0) < \varepsilon$ for each $-1 < x < -1 + \frac{4}{5}\delta + \frac{1}{2}\frac{4}{5}\delta$ except for finitely many times. Therefore, $|\{x \in (-1,-1+\frac{6}{5}\delta) : f(x,0) > \varepsilon\}| < \omega$. Applying the argument above finitely many times we obtain that $|\{x \in (-1,\frac{1}{5}\delta) : f(x,0) > \varepsilon\}| < \omega$. Similarly, starting from the right end of the segment $[-1,1]$, we can prove that $|\{x \in (\frac{-1}{5}\delta,1) : f(x,0) < 2\varepsilon\}| < \omega$. This contradiction completes the proof of the claim. □

Now we shall construct a space X. Consider the Stone-Čech extension βY of the space Y.

Let

$$\begin{aligned}\tilde G_1 &= \mathrm{Cl}_{\beta Y}(F_1) \setminus Y\\ \tilde G_2 &= \mathrm{Cl}_{\beta Y}(F_2) \setminus Y\\ \tilde G_3 &= \beta Y \setminus \big(\tilde G_1 \cup \tilde G_2 \cup Y\big).\end{aligned}$$

Let $X = G_1 \cup G_2 \cup G_3 \cup Y$ be the disjoint union of copies of sets $\tilde G_1, \tilde G_2, \tilde G_3, \tilde Y$. Basic elements for topology on X are either:

1. U for some open $U \subset Y$;
2. $\{g\} \cup (U \cap Y)$, for some $g \in G_3$, and some neighborhood U of g in βY;
3. $\{g\} \cup (U \cap U_1)$, for some $g \in G_1$, and some neighborhood U of g in βY;
4. $\{g\} \cup (U \cap U_2)$, for some $g \in G_2$, and some neighborhood U of g in βY.

Now $U_1 \cap U_2 = \varnothing$ implies that X is a Hausdorff space. Clearly, every open cover γ of X induces an open cover γ' of βY, members of which are unions of at most two elements of γ. It follows that Y is relatively compact in X. Finally, $\mathrm{Cl}_X F_1 = G_1$, $\mathrm{Cl}_X F_2 = G_2$ yields $\mathrm{Cl}_X F_1 \cap \mathrm{Cl}_X F_2 = \varnothing$. Thus Y and X satisfy all the required conditions.

We now turn to the second example.

1.10 Example. *There is a regular non-Tychonoff space $\tilde Y$ with the countable separation property which is relatively compact in some Hausdorff space.*

Proof: We use the notation of Lemma 1.8. Feed Y and X into the "Jones Machine" ([Jo], see also [PW]). Let $A = F_1$, $B = F_2$, $C = \mathrm{Cl}_X F_1$, $D = \mathrm{Cl}_X F_2$ and let $\bar X$ be the quotient space obtained from $X \times \omega$ by identifying C_{2n+1} with C_{2n+2} and D_{2n+2} with D_{2n+3} for each $n \in \mathbb{N}$ and $q : X \times \omega \to \bar X$ be the natural quotient map. Let $\tilde X = \bar X \cup \{z\}$ topologized as follows: $\bar X$ is an open subspace of $\tilde X$, and $\big\{\{z\} \cup \cup_{n>k} X_n : k \in \omega\big\}$ is a base in z. Let $\tilde Y = q(Y \times \omega) \cup \{z\}$. Clearly, $\tilde X$ is Hausdorff and $\tilde Y$ is a regular, non-Tychonoff subspace of $\tilde X$ ([Jo], see also

[PW]). Since for each $n \in \omega$, $Y \times n$ is relatively compact in $X \times n$ and hence in $\tilde{X}$ and every neighborhood of z contains all except at most finitely many $Y \times n$, $\tilde{Y}$ is relatively compact in $\tilde{X}$. Finally, since $\tilde{Y} \setminus \{z\}$ is Tychonoff and $F_1 = \cap_{i\in\omega} W_i$ where $\mathrm{Cl}_Y\, W_{i+1} \subset W_i$, it follows that Y has the countable separation property. □

If we use the "Double Jones Machine" instead of the "Jones Machine" in Example 1.10 (i.e. consider the factor space of the product $X \times \mathbb{Z}$ and add two points $-\infty$ and ∞) we obtain a bit stronger

1.11 Example. *There is a regular space Z which is relatively compact in some Hausdorff space and has the countable separation property, but which is not functionally Hausdorff.*

Now using Herrlich technique [He] one can obtain

1.12 Example. *There is a regular space Z which is relatively compact in some Hausdorff space, such that all real-valued continuous functions on Z are constants.*

What if X has some separation property stronger than Hausdorff? First, since every Hausdorff space can be embedded as a closed subspace into some semiregular space the following assertion holds.

2.1 Proposition. *A space Y can be embedded as a relatively compact subspace into a Hausdorff space if and only if Y can be embedded as a relatively compact subspace into a semiregular space.*

On the other hand, if Y is relatively compact in some Urysohn space, then Y must be Tychonoff. Indeed, we have

2.2 Theorem. *Let Y be a dense relatively compact subspace of an Urysohn space X. Then there is a compact space Z, and condensation $f : X \to Z$, such that for each $y \in X$ the restriction $f|_{Y\cup\{y\}}$ of f to $Y \cup \{y\}$ is a homeomorphism of $Y \cup \{y\}$ onto the image.*

To prove this we need the following

2.3 Proposition. *Let Y be a dense relatively compact subspace of a space X. Then X is H-closed.*

Proof: Direct check. □

Proof of the theorem: Let Z be the semiregularization of a space Z, and let $f : X \to Z$ be the natural condensation. Then Z is a semiregular Urysohn space, and $f(Y)$ is relatively compact in Z. Proposition 2.3 yields that Z is H-closed. So Z is a semiregular Urysohn H-closed space. Hence Z is compact. Now take arbitrary $y \in X$. Then $Y \cup \{y\}$ is relatively compact in X. Therefore, the semiregularization of $Y \cup \{y\}$ is again $Y \cup \{y\}$. It follows that $f|_{Y\cup\{y\}}$ is a homeomorphism. □

2.4 Definition. A subspace Y of a space X is said to be *real-normal* in X iff every two subspaces of Y having disjoint closures in X can be separated in X by a continuous real-valued function.

2.5 Corollary. *Let Y be a dense relatively compact subspace of an Urysohn space X. Then Y is Tychonoff, X is functionally Hausdorff and Y is real-normal in X.*

PROOF: We shall prove that Y is real-normal in X, other properties are obvious. Let $F_1, F_2 \subset Y$, $\mathrm{Cl}_X F_1 \cap \mathrm{Cl}_X F_2 = \varnothing$. Use the notations of 2.2. Since for each $y \in X$ $f|_{Y\cup\{y\}}$ is a homeomorphism $\mathrm{Cl}_Z f(F_1) \cap \mathrm{Cl}_Z f(F_2) = \varnothing$. Hence $\mathrm{Cl}_Z f(F_1)$ and $\mathrm{Cl}_Z f(F_2)$ can be separated in compact space Z by a continuous real-valued function. Therefore, the same is true for F_1 and F_2 in X. $\square$

2.6 Problem. *Find an "inner" characterization of the potential compactness.*

2.7 Problem. *Is there a regular space without a dense Tychonoff subspace?*

In the opinion of the authors, the last question should have the negative answer, but to construct the corresponding example entirely new ideas will be needed. Indeed, regular non-Tychonoff spaces are always constructed by adding new "bad" points to a "not-so-nice" Tychonoff space.

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