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Topological games and product spaces

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Abstract: In this paper, we deal with the product of spaces which are either $\mathcal{G}$-spaces or $\mathcal{G}_p$-spaces, for some $p \in \omega^*$. These spaces are defined in terms of a two-person infinite game over a topological space. All countably compact spaces are $\mathcal{G}$-spaces, and every $\mathcal{G}_p$-space is a $\mathcal{G}$-space, for every $p \in \omega^*$. We prove that if $\{X_\mu : \mu < \omega_1\}$ is a set of spaces whose product $X = \prod_{\mu<\omega_1} X_\mu$ is a $\mathcal{G}$-space, then there is $A \in [\omega_1]^{\leq\omega}$ such that X_μ is countably compact for every $\mu \in \omega_1 \setminus A$. As a consequence, X^{ω_1} is a $\mathcal{G}$-space iff X^{ω_1} is countably compact, and if $X^{2^{\mathfrak{c}}}$ is a $\mathcal{G}$-space, then all powers of X are countably compact. It is easy to prove that the product of a countable family of $\mathcal{G}_p$ spaces is a $\mathcal{G}_p$-space, for every $p \in \omega^*$. For every $1 \leq n < \omega$, we construct a space X such that X^n is countably compact and X^{n+1} is not a $\mathcal{G}$-space. If $p, q \in \omega^*$ are RK-incomparable, then we construct a $\mathcal{G}_p$-space X and a $\mathcal{G}_q$-space Y such that $X \times Y$ is not a $\mathcal{G}$-space. We give an example of two free ultrafilters p and q on ω such that $p <_{RK} q$, p and q are RF-incomparable, $p \approx_C q$ ($\leq_C$ is the *Comfort* order on ω^*) and there are a $\mathcal{G}_p$-space X and a $\mathcal{G}_q$-space Y whose product $X \times Y$ is not a $\mathcal{G}$-space.

Keywords: RF-order, RK-order, Comfort-order, p-limit, p-compact, $\mathcal{G}$-space, $\mathcal{G}_p$-space, countably compact
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