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An alternative way to classify some Generalized Elliptic Curves and their isotopic loops

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Abstract: The Generalized Elliptic Curves ($GECs$) are pairs (Q, T), where T is a family of triples (x, y, z) of "points" from the set Q characterized by equalities of the form $x.y = z$, where the law $x.y$ makes Q into a totally symmetric quasigroup. Isotopic loops arise by setting $x * y = u.(x.y)$. When $(x.y).(a.b) = (x.a).(y.b)$, identically (Q, T) is an entropic GEC and $(Q, *)$ is an abelian group. Similarly, a terentropic GEC may be characterized by $x^2.(a.b) = (x.a)(x.b)$ and $(Q, *)$ is then a Commutative Moufang Loop (CML). If in addition $x^2 = x$, we have Hall $GECs$ and $(Q, *)$ is an exponent 3 CML. Any finite terentropic GEC admits a direct decomposition in primary components and only the 3-component may eventually be non entropic, in which case its order is at least 81. It turns out that there are fifteen order 81 terentropic $GECs$ (including just three non-entropic $GECs$). In class 2 $CMLs$ the associator enjoys some pseudo-linearity: $(x*x', y, z) = (x, y, z)*(x', y, z)$. We are thus led to searching representatives in the set $AT(n, m, K)$ of image-rank m alternate trilinear mappings from $(V(n, K))^3$ to $V(m, K)$ up to changes of basis in these K-vector spaces. Denote by $\alpha(n, m, K)$ the cardinal number of the sets of representatives. We establish that $\alpha(5, 2, K) \leq 5$ whenever each field-element is quadratic; moreover $\alpha(5, 2, \mathbb{F}_3) = 6$ and $\alpha(6, 2, \mathbb{F}_3) \geq 13$. We obtained a transfer theorem providing a one-to-one correspondence between the classes from $AT(n, m, \mathbb{F}_3)$ and the rank $n+1$ class 2 Hall $GECs$ of 3-order $n+m$. Now $\alpha(7, 1, GF(3^s)) = 11$ for any s. We derive a complete classification and explicit descriptions of the eleven Hall $GECs$ whose rank and 3-order both equal 8. One of these has for automorphism group some extension of the Chevalley group $G_2(\mathbb{F}_3)$.

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