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Spaces of continuous functions, Σ-products and Box Topology

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Abstract: For a Tychonoff space X, we will denote by X_0 the set of its isolated points and X_1 will be equal to $X \setminus X_0$. The symbol $C(X)$ denotes the space of real-valued continuous functions defined on X. $\Box\mathbb{R}^\kappa$ is the Cartesian product $\mathbb{R}^\kappa$ with its box topology, and $C_\Box(X)$ is $C(X)$ with the topology inherited from $\Box\mathbb{R}^X$. By $\widehat{C}(X_1)$ we denote the set $\{f \in C(X_1) : f$ can be continuously extended to all of $X\}$. A space X is almost-ω-resolvable if it can be partitioned by a countable family of subsets in such a way that every non-empty open subset of X has a non-empty intersection with the elements of an infinite subcollection of the given partition. We analyze $C_\Box(X)$ when X_0 is F_σ and prove: (1) for every topological space X, if X_0 is F_σ in X, and $\emptyset \neq X_1 \subset cl_X X_0$, then $C_\Box(X) \cong \Box\mathbb{R}^{X_0}$; (2) for every space X such that X_0 is F_σ, $cl_X X_0 \cap X_1 \neq \emptyset$, and $X_1 \setminus cl_X X_0$ is almost-ω-resolvable, then $C_\Box(X)$ is homeomorphic to a free topological sum of $\leq |\widehat{C}(X_1)|$ copies of $\Box\mathbb{R}^{X_0}$, and, in this case, $C_\Box(X) \cong \Box\mathbb{R}^{X_0}$ if and only if $|\widehat{C}(X_1)| \leq 2^{|X_0|}$. We conclude that for a space X such that X_0 is F_σ, $C_\Box(X)$ is never normal if $|X_0| > \aleph_0$ [La], and, assuming CH, $C_\Box(X)$ is paracompact if $|X_0| = \aleph_0$ [Ru2]. We also analyze $C_\Box(X)$ when $|X_1| = 1$ and when X is countably compact, and we scrutinize under what conditions $\Box\mathbb{R}^\kappa$ is homeomorphic to some of its "Σ-products"; in particular, we prove that $\Box\mathbb{R}^\omega$ is homeomorphic to each of its subspaces $\{f \in \Box\mathbb{R}^\omega : \{n \in \omega : f(n) = 0\} \in p\}$ for every $p \in \omega^*$, and it is homeomorphic to $\{f \in \Box\mathbb{R}^\omega : \forall\epsilon > 0\{n \in \omega : |f(n)| < \epsilon\} \in \mathcal{F}_0\}$ where $\mathcal{F}_0$ is the Fréchet filter on ω.

Keywords: spaces of real-valued continuous functions, box topology, Σ-product, almost-ω-resolvable space
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