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The regular topology on $C(X)$

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Abstract: Hewitt [*Rings of real-valued continuous functions. I.*, Trans. Amer. Math. Soc. **64** (1948), 45–99] defined the m-topology on $C(X)$, denoted $C_m(X)$, and demonstrated that certain topological properties of X could be characterized by certain topological properties of $C_m(X)$. For example, he showed that X is pseudocompact if and only if $C_m(X)$ is a metrizable space; in this case the m-topology is precisely the topology of uniform convergence. What is interesting with regards to the m-topology is that it is possible, with the right kind of space X, for $C_m(X)$ to be highly non-metrizable. E. van Douwen [*Nonnormality of spaces of real functions*, Topology Appl. **39** (1991), 3–32] defined the class of DRS-spaces and showed that if X was such a space, then $C_m(X)$ satisfied the property that all countable subsets of $C_m(X)$ are closed. In J. Gomez-Perez and W.Wm. McGovern, *The m-topology on $C_m(X)$ revisited*, Topology Appl. **153**, (2006), no. 11, 1838–1848, the authors demonstrated the converse, completing the characterization. In this article we define a finer topology on $C(X)$ based on positive regular elements. It is the authors' opinion that the new topology is a more well-behaved topology with regards to passing from $C(X)$ to $C^*(X)$. In the first section we compute some common cardinal invariants of the preceding space $C_r(X)$. In Section 2, we characterize when $C_r(X)$ satisfies the property that all countable subsets are closed. We call such a space for which this happens a weak DRS-space and demonstrate that X is a weak DRS-space if and only if βX is a weak DRS-space. This is somewhat surprising as a DRS-space cannot be compact. In the third section we give an internal characterization of separable weak DRS-spaces and use this to show that a metrizable space is a weak DRS-space precisely when it is nowhere separable.

Keywords: DRS-space, Stone-Čech compactification, rings of continuous functions, $C(X)$
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