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## *Free non-archimedean topological groups*

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**Abstract:** We study free topological groups defined over uniform spaces in some subclasses of the class **NA** of non-archimedean groups. Our descriptions of the corresponding topologies show that for metrizable uniformities the corresponding free balanced, free abelian and free Boolean **NA** groups are also metrizable. Graev type ultra-metrics determine the corresponding free topologies. Such results are in a striking contrast with free balanced and free abelian topological groups cases (in standard varieties). Another contrasting advantage is that the induced topological group actions on free abelian **NA** groups frequently remain continuous. One of the main applications is: any epimorphism in the category **NA** must be dense. Moreover, the same methods improve the following result of T.H. Fay [*A note on Hausdorff groups*, Bull. Austral. Math. Soc. **13** (1975), 117–119]: the inclusion of a proper open subgroup  $H \hookrightarrow G \in \mathbf{TGR}$  is not an epimorphism in the category **TGR** of all Hausdorff topological groups. A key tool in the proofs is Pestov’s test of epimorphisms [V.G. Pestov, *Epimorphisms of Hausdorff groups by way of topological dynamics*, New Zealand J. Math. **26** (1997), 257–262]. Our results provide a convenient way to produce surjectively universal **NA** abelian and balanced groups. In particular, we unify and strengthen some recent results of Gao [*Graev ultrametrics and surjectively universal non-Archimedean Polish groups*, Topology Appl. **160** (2013), no. 6, 862–870] and Gao-Xuan [*On non-Archimedean Polish groups with two-sided invariant metrics*, preprint, 2012] as well as classical results about profinite groups which go back to Iwasawa and Gildenhuys-Lim [*Free pro- $C$ -groups*, Math. Z. **125** (1972), 233–254].

**Keywords:** epimorphisms, free profinite group, free topological  $G$ -group, non-archimedean group, ultra-metric, ultra-norm

**AMS Subject Classification:** 54H11, 22A05, 46S10, 54H15, 54E15

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