

Luděk Zajíček

Remarks on Fréchet differentiability of pointwise Lipschitz, cone-monotone and quasiconvex functions

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Abstract: We present some consequences of a deep result of J. Lindenstrauss and D. Preiss on Γ -almost everywhere Fréchet differentiability of Lipschitz functions on c_0 (and similar Banach spaces). For example, in these spaces, every continuous real function is Fréchet differentiable at Γ -almost every x at which it is Gâteaux differentiable. Another interesting consequences say that both cone-monotone functions and continuous quasiconvex functions on these spaces are Γ -almost everywhere Fréchet differentiable. In the proofs we use a general observation that each version of the Rademacher theorem for real functions on Banach spaces (i.e., a result on a.e. Fréchet or Gâteaux differentiability of Lipschitz functions) easily implies by a method of J. Malý a corresponding version of the Stepanov theorem (on a.e. differentiability of pointwise Lipschitz functions). Using the method of separable reduction, we extend some results to several non-separable spaces.

Keywords: cone-monotone function; Fréchet differentiability; Gâteaux differentiability; pointwise Lipschitz function; Γ -null set; quasiconvex function; separable reduction

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