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Differences of two semiconvex functions on the real line

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Abstract: It is proved that real functions on $\mathbb{R}$ which can be represented as the difference of two semiconvex functions with a general modulus (or of two lower C^1-functions, or of two strongly paraconvex functions) coincide with semismooth functions on $\mathbb{R}$ (i.e. those locally Lipschitz functions on $\mathbb{R}$ for which $f'_+(x) = \lim_{t \to x+} f'_+(t)$ and $f'_-(x) = \lim_{t \to x-} f'_-(t)$ for each x). Further, for each modulus ω, we characterize the class DSC_ω of functions on $\mathbb{R}$ which can be written as $f = g - h$, where g and h are semiconvex with modulus $C\omega$ (for some $C > 0$) using a new notion of $[\omega]$-variation. We prove that $f \in DSC_\omega$ if and only if f is continuous and there exists $D > 0$ such that f'_+ has locally finite $[D\omega]$-variation. This result is proved via a generalization of the classical Jordan decomposition theorem which characterizes the differences of two ω-nondecreasing functions (defined by the inequality $f(y) \geq f(x) - \omega(y - x)$ for $y > x$) on $[a, b]$ as functions with finite $[2\omega]$-variation. The research was motivated by a recent article by J. Duda and L. Zajíček on Gâteaux differentiability of semiconvex functions, in which surfaces described by differences of two semiconvex functions naturally appear.

Keywords: semiconvex function with general modulus; difference of two semiconvex functions; ω-nondecreasing function; $[\omega]$-variation; regulated function
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