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Structure theory for the group algebra of the symmetric group, with applications to polynomial identities for the octonions

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Abstract: This is a survey paper on applications of the representation theory of the symmetric group to the theory of polynomial identities for associative and nonassociative algebras. In §1, we present a detailed review (with complete proofs) of the classical structure theory of the group algebra $\mathbb{F}S_n$ of the symmetric group S_n over a field $\mathbb{F}$ of characteristic 0 (or $p > n$). The goal is to obtain a constructive version of the isomorphism $\psi\colon \bigoplus_\lambda M_{d_\lambda}(\mathbb{F}) \longrightarrow \mathbb{F}S_n$ where λ is a partition of n and d_λ counts the standard tableaux of shape λ. Young showed how to compute ψ; to compute its inverse, we use an efficient algorithm for representation matrices discovered by Clifton. In §2, we discuss constructive methods based on §1 which allow us to analyze the polynomial identities satisfied by a specific (non)associative algebra: fill and reduce algorithm, module generators algorithm, Bondari's algorithm for finite dimensional algebras. In §3, we study the multilinear identities satisfied by the octonion algebra $\mathbb{O}$ over a field of characteristic 0. For $n \leq 6$ we compare our computational results with earlier work of Racine, Hentzel & Peresi, Shestakov & Zhukavets. Going one step further, we verify computationally that every identity in degree 7 is a consequence of known identities of lower degree; this result is our main original contribution. This gap (no new identities in degree 7) motivates our concluding conjecture: the known identities for $n \leq 6$ generate all of the octonion identities in characteristic 0.

Keywords: symmetric group; group algebra; Young diagrams; standard tableaux; idempotents; matrix units; two-sided ideals; Wedderburn decomposition; representation theory; Clifton's algorithm; computer algebra; polynomial identities; nonassociative algebra; octonions

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