Václav Kryštof
Generalized versions of Ilmanen lemma: Insertion of $C^{1,\omega}$ or $C^{1,\omega}_{\mathrm{loc}}$ functions

Abstract: We prove that for a normed linear space X, if $f_1\colon X \to \mathbb{R}$ is continuous and semiconvex with modulus ω, $f_2\colon X \to \mathbb{R}$ is continuous and semiconcave with modulus ω and $f_1 \le f_2$, then there exists $f \in C^{1,\omega}(X)$ such that $f_1 \le f \le f_2$. Using this result we prove a generalization of Ilmanen lemma (which deals with the case $\omega(t) = t$) to the case of an arbitrary nontrivial modulus ω. This generalization (where a $C^{1,\omega}_{\mathrm{loc}}$ function is inserted) gives a positive answer to a problem formulated by A. Fathi and M. Zavidovique in 2010.

REFERENCES

[1] Bernard P., *Lasry-Lions regularization and a lemma of Ilmanen*, Rend. Semin. Mat. Univ. Padova **124** (2010), 221–229.

[2] Cannarsa P., Sinestrari C., *Semiconcave Functions, Hamilton-Jacobi Equations, and Optimal Control*, Progress in Nonlinear Differential Equations and Their Applications, 58, Birkhäuser, Boston, 2004.

[3] Clarke F. H., Ledyaev Yu. S., Stern R. J., Wolenski P. R., *Nonsmooth Analysis and Control Theory*, Graduate Texts in Mathematics, 178, Springer, New York, 1998.

[4] Duda J., Zajíček L., *Semiconvex functions: representations as suprema of smooth functions and extensions*, J. Convex Anal. **16** (2009), no. 1, 239–260.

[5] Duda J., Zajíček L., *Smallness of singular sets of semiconvex functions in separable Banach spaces*, J. Convex Anal. **20** (2013), no. 2, 573–598.

[6] Fathi A., Figalli A., *Optimal transportation on non-compact manifolds*, Israel J. Math. **175** (2010), 1–59.

[7] Fathi A., Zavidovique M., *Ilmanen's lemma on insertion of $C^{1,1}$ functions*, Rend. Semin. Mat. Univ. Padova **124** (2010), 203–219.

[8] Hájek P., Johanis M., *Smooth Analysis in Banach Spaces*, De Gruyter Series in Nonlinear Analysis and Applications, 19, De Gruyter, Berlin, 2014.

[9] Ilmanen T., *The level-set flow on a manifold*, Differential Geometry: Partial Differential Equations on Manifolds (Los Angeles, CA, 1990), Proc. Sympos. Pure Math., 54, Part 1, Amer. Math. Soc., Providence, 1993, pp. 193–204.

[10] Jourani A., Thibault L., Zagrodny D., *$C^{1,\omega(\cdot)}$-regularity and Lipschitz-like properties of subdifferential*, Proc. Lond. Math. Soc. (3) **105** (2012), no. 1, 189–223.

[11] Koc M., Kolář J., *Extensions of vector-valued functions with preservation of derivatives*, J. Math. Anal. Appl. **449** (2017), no. 1, 343–367.

[12] Kryštof V., *Semiconvex Functions and Their Differences*, Master Thesis, Charles University, Praha, 2016 (Czech).

[13] Rolewicz S., *On $\alpha(\cdot)$-paraconvex and strongly $\alpha(\cdot)$-paraconvex functions*, Control Cybernet. **29** (2000), no. 1, 367–377.

[14] Rolewicz S., *On the coincidence of some subdifferentials in the class of $\alpha(\cdot)$-paraconvex functions*, Optimization **50** (2001), no. 5–6, 353–360.

[15] Toruńczyk H., *Smooth partitions of unity on some non-separable Banach spaces*, Studia Math. **46** (1973), 43–51.