Haosui Duanmu, William Weiss

Finitely-additive, countably-additive and internal probability measures

Comment.Math.Univ.Carolin. 59,4 (2018) 467 –485.

Abstract: We discuss two ways to construct standard probability measures, called push-down measures, from internal probability measures. We show that the Wasserstein distance between an internal probability measure and its push-down measure is infinitesimal. As an application to standard probability theory, we show that every finitely-additive Borel probability measure P on a separable metric space is a limit of a sequence of countably-additive Borel probability measures $\{P_n\}_{n\in\mathbb{N}}$ in the sense that $\int f\,\mathrm{d}P = \lim_{n\to\infty} \int f\,\mathrm{d}P_n$ for all bounded uniformly continuous real-valued function f if and only if the space is totally bounded.

Keywords: nonstandard model in mathematics; nonstandard analysis; nonstandard measure theory; convergence of probability measures

AMS Subject Classification: 03H05, 26E35, 28E05, 60B10

References

[1] Aldaz J. M., *Representation of Measures via the Standard Part Map*, Ph.D. Thesis, University of Illinois at Urbana-Champaign, Champaign, 1991.

[2] Aldaz J. M., *Representing abstract measures by Loeb measures: a generalization of the standard part map*, Proc. Amer. Math. Soc. **123** (1995), no. 9, 2799–2808.

[3] Anderson R. M., *Star-finite representations of measure spaces*, Trans. Amer. Math. Soc. **271** (1982), no. 2, 667–687.

[4] Anderson R. M., Rashid S., *A nonstandard characterization of weak convergence*, Proc. Amer. Math. Soc. **69** (1978), no. 2, 327–332.

[5] Arkeryd L. O., Cutland N. J., Henson C. W., eds., *Nonstandard Analysis*, Theory and Applications, Proc. of the NATO Advanced Study Institute on Nonstandard Analysis and Its Applications, Edinburgh, 1996, NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences, 493, Kluwer Academic Publishers Group, Dordrecht, 1997.

[6] Bhaskara R. K. P. S., Bhaskara R. M., *Theory of Charges*, A study of finitely additive measures, Pure and Applied Mathematics, 109, Academic Press, Harcourt Brace Jovanovich Publishers, New York, 1983.

[7] Cutland N. J., Neves V., Oliveira F., Sousa-Pinto J., eds., *Developments in Nonstandard Mathematics*, Pitman Research Notes in Mathematics Series, 336, Papers from the Int. Col. (CIMNS94), Aveiro, 1994, Longman, Harlow, 1995.

[8] Duanmu H., Rosenthal J. S., Weiss W., *Ergodicity of Markov Processes via Non-standard Analysis*, available at `http://probability.ca/jeff/ftpdir/duanmu1.pdf`.

[9] Duanmu H., Roy D. M., *On Extended Admissible Procedures and Their Nonstandard Bayes Risk*, available at arXiv:1612.09305v2 [math.ST].

[10] Kadane J. B., Schervish M. J., Seidenfeld T., *Statistical implications of finitely additive probability*, Stud. Bayesian Econometrics Statist., 6, North-Holland, Amsterdam, 1986, pages 59–76.

[11] Keisler H. J., *An infinitesimal approach to stochastic analysis*, Mem. Amer. Math. Soc. **48** (1984), no. 297, 184 pages.

[12] Lindström T., *Pushing Down Loeb-Measures*, Pure Mathematics: Preprint Series, Matematisk Institutt Universitetet i Oslo, Oslo, 1981.

[13] Loeb P. A., *Conversion from nonstandard to standard measure spaces and applications in probability theory*, Trans. Amer. Math. Soc. **211** (1975), 113–122.

[14] McShane E. J., *Extension of range of functions*, Bull. Amer. Math. Soc. **40** (1934), no. 12, 837–842.

[15] Render H., *Pushing down Loeb measures*, Math. Scand. **72** (1993), no. 1, 61–84.

[16] Rosenthal J. S., *A First Look at Rigorous Probability Theory*, World Scientific Publishing, Hackensack, 2006.

[17] Ross D., *Compact measures have Loeb preimages*, Proc. Amer. Math. Soc. **115** (1992), no. 2, 365–370.

[18] Ross D. A., *Pushing down infinite Loeb measures*, Math. Scand. **104** (2009), no. 1, 108–116.
[19] Yosida K., Hewitt E., *Finitely additive measures*, Trans. Amer. Math. Soc. **72** (1952), 46–66.