Ondřej Zindulka

Strong measure zero and meager-additive sets through the prism of fractal measures

Comment.Math.Univ.Carolin. 60,1 (2019) 131 –155.

Abstract: We develop a theory of sharp measure zero sets that parallels Borel's strong measure zero, and prove a theorem analogous to Galvin–Mycielski–Solovay theorem, namely that a set of reals has sharp measure zero if and only if it is meager-additive. Some consequences: A subset of 2^ω is meager-additive if and only if it is $\mathcal{E}$-additive; if $f\colon 2^\omega \to 2^\omega$ is continuous and X is meager-additive, then so is $f(X)$.

Keywords: meager-additive; $\mathcal{E}$-additive; strong measure zero; sharp measure zero; Hausdorff dimension; Hausdorff measure
AMS Subject Classification: 03E05, 03E20, 28A78

References

[1] Bartoszyński T., Judah H., *Set Theory*, On the structure of the real line. A K Peters, Wellesley, 1995.
[2] Bartoszyński T., Shelah S., *Closed measure zero sets*, Ann. Pure Appl. Logic **58** (1992), no. 2, 93–110.
[3] Besicovitch A. S., *Concentrated and rarified sets of points*, Acta Math. **62** (1933), no. 1, 289–300.
[4] Besicovitch A. S., *Correction*, Acta Math. **62** (1933), no. 1, 317–318.
[5] Borel E., *Sur la classification des ensembles de mesure nulle*, Bull. Soc. Math. France **47** (1919), 97–125 (French).
[6] Carlson T. J., *Strong measure zero and strongly meager sets*, Proc. Amer. Math. Soc. **118** (1993), no. 2, 577–586.
[7] Corazza P., *The generalized Borel conjecture and strongly proper orders*, Trans. Amer. Math. Soc. **316** (1989), no. 1, 115–140.
[8] Federer H., *Geometric Measure Theory*, Die Grundlehren der mathematischen Wissenschaften, 153, Springer, New York, 1969.
[9] Fremlin D. H., *Measure Theory. Vol. 5, Set-theoretic Measure Theory, Part I*, Torres Fremlin, Colchester, 2015.
[10] Galvin F., Miller A. W., *γ-sets and other singular sets of real numbers*, Topology Appl. **17** (1984), no. 2, 145–155.
[11] Galvin F., Mycielski J., Solovay R. M., *Strong measure zero sets*, Abstract 79T–E25, Not. Am. Math. Soc. **26** (1979), A-280.
[12] Galvin F., Mycielski J., Solovay R. M., *Strong measure zero and infinite games*, Arch. Math. Logic **56** (2017), no. 7–8, 725–732.
[13] Gerlits J., Nagy Z., *Some properties of $C(X)$ I*, Topology Appl. **14** (1982), no. 2, 151–161.
[14] Gödel K., *The consistency of the axiom of choice and of the generalized continuum-hypothesis*, Proc. Natl. Acad. Sci. USA **24** (1938), no. 12, 556–557.
[15] Gödel K., *The Consistency of the Continuum Hypothesis*, Annals of Mathematics Studies, 3, Princeton University Press, Princeton, 1940.
[16] Howroyd J. D., *On the Theory of Hausdorff Measures in Metric Spaces*, Ph.D. Thesis, University College, London, 1994.
[17] Howroyd J. D., *On Hausdorff and packing dimension of product spaces*, Math. Proc. Cambridge Philos. Soc. **119** (1996), no. 4, 715–727.
[18] Hrušák M., Wohofsky W., Zindulka O., *Strong measure zero in separable metric spaces and Polish groups*, Arch. Math. Logic **55** (2016), no. 1–2, 105–131.
[19] Hrušák M., Zapletal J., *Strong measure zero sets in Polish groups*, Illinois J. Math. **60** (2016), no. 3–4, 751–760.
[20] Kelly J. D., *A method for constructing measures appropriate for the study of Cartesian products*, Proc. London Math. Soc. (3) **26** (1973), 521–546.
[21] Kysiak M., *On Erdős-Sierpiński Duality between Lebesgue Measure and Baire Category*, Master's Thesis, Uniwersytet Warszawski, Warszawa, 2000 (Polish).

[22] Laver R., *On the consistency of Borel's conjecture*, Acta Math. **137** (1976), no. 3–4, 151–169.
[23] Munroe M. E., *Introduction to Measure and Integration*, Addison-Wesley Publishing Company, Cambridge, 1953.
[24] Nowik A., Scheepers M., Weiss T., *The algebraic sum of sets of real numbers with strong measure zero sets*, J. Symbolic Logic **63** (1998), no. 1, 301–324.
[25] Nowik A., Weiss T., *On the Ramseyan properties of some special subsets of* 2^ω *and their algebraic sums*, J. Symbolic Logic **67** (2002), no. 2, 547–556.
[26] Pawlikowski J., *A characterization of strong measure zero sets*, Israel J. Math. **93** (1996), 171–183.
[27] Rogers C. A., *Hausdorff Measures*, Cambridge University Press, London, 1970.
[28] Scheepers M., *Finite powers of strong measure zero sets*, J. Symbolic Logic **64** (1999), no. 3, 1295–1306.
[29] Shelah S., *Every null-additive set is meager-additive*, Israel J. Math. **89** (1995), no. 1–3, 357–376.
[30] Sierpiński W., *Sur un ensemble non denombrable, dont toute image continue est de mesure nulle*, Fundamenta Mathematicae **11** (1928), no. 1, 302–304 (French).
[31] Tsaban B., Weiss T., *Products of special sets of real numbers*, Real Anal. Exchange **30** (2004/05), no. 2, 819–835.
[32] Weiss T., *On meager additive and null additive sets in the Cantor space* 2^ω *and in* $\mathbb{R}$, Bull. Pol. Acad. Sci. Math. **57** (2009), no. 2, 91–99.
[33] Weiss T., *Addendum to "On meager additive and null additive sets in the Cantor space* 2^ω *and in* $\mathbb{R}$*"* (Bull. Polish Acad. Sci. Math. **57** (2009), 91–99), Bull. Pol. Acad. Sci. Math. **62** (2014), no. 1, 1–9.
[34] Weiss T., *Properties of the intersection ideal* $\mathcal{M} \cap \mathcal{N}$ *revisited*, Bull. Pol. Acad. Sci. Math. **65** (2017), no. 2, 107–111.
[35] Weiss T., Tsaban B., *Topological diagonalizations and Hausdorff dimension*, Note Mat. **22** (2003/04), no. 2, 83–92.
[36] Wohofsky W., *Special Sets of Real Numbers and Variants of the Borel Conjecture*, Ph.D. Thesis, Technische Universität Wien, Wien, 2013.
[37] Zakrzewski P., *Universally meager sets*, Proc. Amer. Math. Soc. **129** (2001), no. 6, 1793–1798.
[38] Zakrzewski P., *Universally meager sets. II*, Topology Appl. **155** (2008), no. 13, 1445–1449.
[39] Zindulka O., *Universal measure zero, large Hausdorff dimension, and nearly Lipschitz maps*, Fund. Math. **218** (2012), no. 2, 95–119.
[40] Zindulka O., *Packing measures and dimensions on Cartesian products*, Publ. Mat. **57** (2013), no. 2, 393–420.