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Measure-geometric Laplacians for partially atomic measures

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Abstract: Motivated by the fundamental theorem of calculus, and based on the works of W. Feller as well as M. Kac and M. G. Kreĭn, given an atomless Borel probability measure η supported on a compact subset of $\mathbb{R}$ U. Freiberg and M. Zähle introduced a measure-geometric approach to define a first order differential operator ∇_η and a second order differential operator Δ_η, with respect to η. We generalize this approach to measures of the form $\eta := \nu + \delta$, where ν is non-atomic and δ is finitely supported. We determine analytic properties of ∇_η and Δ_η and show that Δ_η is a densely defined, unbounded, linear, self-adjoint operator with compact resolvent. Moreover, we give a systematic way to calculate the eigenvalues and eigenfunctions of Δ_η. For two leading examples, we determine the eigenvalues and the eigenfunctions, as well as the asymptotic growth rates of the eigenvalue counting function.

Keywords: Kreĭn–Feller operator; spectral asymptotics; harmonic analysis
AMS Subject Classification: 47G30, 42B35, 35P20

References

[1] Arzt P., *Measure theoretic trigonometric functions*, J. Fractal Geom. **2** (2015), no. 2, 115–169.

[2] Beals R., Greiner P. C., *Strings, waves, drums: spectra and inverse problems*, Anal. Appl. (Singap.) **7** (2009), no. 2, 131–183.

[3] Berry M. V., *Distribution of modes in fractal resonators*, Structural Stability in Physics, Proc. Internat. Symposia Appl. Catastrophe Theory and Topological Concepts in Phys., Inst. Inform. Sci., Univ. Tübingen, 1978, Springer Ser. Synergetics, 4, Springer, Berlin, 1979, pages 51–53.

[4] Berry M. V., *Some geometric aspects of wave motion: wavefront dislocations, diffraction catastrophes, diffractals*, Geometry of the Laplace operator, Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979, Proc. Sympos. Pure Math., XXXVI, Amer. Math. Soc., Providence, 1980, pages 13–28.

[5] Biggs N., *Algebraic Graph Theory*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1993.

[6] Ehnes T., *Stochastic heat equations defined by fractal Laplacians on Cantor-like sets*, available at arXiv: 1902.02175v2 [math.PR] (2019), 27 pages.

[7] Feller W., *Generalized second order differential operators and their lateral conditions*, Illinois J. Math. **1** (1957), 459–504.

[8] Freiberg U., *A survey on measure geometric Laplacians on Cantor like sets*, Wavelet and fractal methods in science and engineering, Part I., Arab. J. Sci. Eng. Sect. C Theme Issues **28** (2003), no. 1, 189–198.

[9] Freiberg U., *Analytical properties of measure geometric Krein–Feller-operators on the real line*, Math. Nachr. **260** (2003), 34–47.

[10] Freiberg U., *Spectral asymptotics of generalized measure geometric Laplacians on Cantor like sets*, Forum Math. **17** (2005), no. 1, 87–104.

[11] Freiberg U., Zähle M., *Harmonic calculus on fractals—a measure geometric approach. I*, Potential Anal. **16** (2002), no. 3, 265–277.

[12] Fujita T., *A fractional dimension, self-similarity and a generalized diffusion operator*, Probabilistic Methods in Mathematical Physics, Katata/Kyoto, 1985, Academic Press, Boston, 1987, pages 83–90.

[13] Gordon C., Webb D., Wolpert S., *Isospectral plane domains and surfaces via Riemannian orbifolds*, Invent. Math. **110** (1992), no. 1, 1–22.

[14] Halmos P. R., *Measure Theory*, D. Van Nostrand Company, New York, 1950.

[15] Jin X., *Spectral representation of one-dimensional Liouville Brownian motion and Liouville Brownian excursion*, available at arXiv: 1705.01726v1 [math.PR] (2017), 23 pages.

[16] Kac I. S., Kreĭn M. G., *Criteria for the discreteness of the spectrum of a singular string*, Izv. Vysš. Učebn. Zaved. Matematika **1958** (1958), no. 2 (3), 136–153.

[17] Kac M., *Can one hear the shape of a drum?*, Amer. Math. Monthly **73** (1966), no. 4, part II, 1–23.

[18] Kesseböhmer M., Niemann A., Samuel T., Weyer H., *Generalised Kreĭn–Feller operators and Liouville Brownian motion via transformations of measure spaces*, available at arXiv:1909.08832v2 [math.FA], (2019), 13 pages.

[19] Kesseböhmer M., Samuel T., Weyer H., *A note on measure-geometric Laplacians*, Monatsh. Math. **181** (2016), no. 3, 643–655.

[20] Kesseböhmer M., Samuel T., Weyer H., *Measure-geometric Laplacians for discrete distributions*, Horizons of Fractal Geometry and Complex Dimensions, Contemp. Math., 731, Amer. Math. Soc., Providence, 2019, pages 133–142.

[21] Kigami J., Lapidus M. L., *Self-similarity of volume measures for Laplacians on p.c.f. self-similar fractals*, Comm. Math. Phys. **217** (2001), no. 1, 165–180.

[22] Kotani S., Watanabe S., *Kreĭn's spectral theory of strings and generalized diffusion processes*, Functional analysis in Markov Processes, Katata/Kyoto, 1981, Lecture Notes in Math., 923, Springer, Berlin, 1982, pages 235–259.

[23] Lapidus M. L., *Fractal drum, inverse spectral problems for elliptic operators and a partial resolution of the Weyl–Berry conjecture*, Trans. Amer. Math. Soc. **325** (1991), no. 2, 465–529.

[24] Lapidus M. L., *Vibrations of fractal drums, the Riemann hypothesis, waves in fractal media and the Weyl–Berry conjecture*, Ordinary and Partial Differential Equations, Vol. IV, Dundee, 1992, Pitman Res. Notes Math. Ser., 289, Longman Sci. Tech., Harlow, 1993, pages 126–209.

[25] Lapidus M. L., Pomerance C., *Fonction zêta de Riemann et conjecture de Weyl–Berry pour les tambours fractals*, C. R. Acad. Sci. Paris Sér. I Math. **310** (1990), no. 6, 343–348 (French. English summary).

[26] Lapidus M. L., Pomerance C., *The Riemann zeta-function and the one-dimensional Weyl–Berry conjecture for fractal drums*, Proc. London Math. Soc. (3) **66** (1993), no. 1, 41–69.

[27] Lapidus M. L., Pomerance C., *Counterexamples to the modified Weyl–Berry conjecture on fractal drums*, Math. Proc. Cambridge Philos. Soc. **119** (1996), no. 1, 167–178.

[28] Milnor J., *Eigenvalues of the Laplace operator on certain manifolds*, Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 542.

[29] Reed M., Simon B., *Methods of Modern Mathematical Physics. I*, Functional analysis, Academic Press, Harcourt Brace Jovanovich Publishers, New York, 1980.

[30] Rhodes R., Vargas V., *Spectral dimension of Liouville quantum gravity*, Ann. Henri Poincaré **15** (2014), no. 12, 2281–2298.

[31] Urakawa H., *Bounded domains which are isospectral but not congruent*, Ann. Sci. École Norm. Sup. (4) **15** (1982), no. 3, 441–456.

[32] Weyl H., *Über die Abhängigkeit der Eigenschwingungen einer Membran und deren Begrenzung*, J. Reine Angew. Math. **141** (1912), 1–11 (German).