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Constructing and embedding mutually orthogonal Latin squares: reviewing both new and existing results

Comment.Math.Univ.Carolin. 61,4 (2020) 437 –457.

Abstract: We review results for the embedding of orthogonal partial Latin squares in orthogonal Latin squares, comparing and contrasting these with results for embedding partial Latin squares in Latin squares. We also present a new construction that uses the existence of a set of t mutually orthogonal Latin squares of order n to construct a set of $2t$ mutually orthogonal Latin squares of order n^t.

Keywords: embedding; mutually orthogonal Latin square
AMS Subject Classification: 05B15

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