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Structural aspects of truncated archimedean vector lattices: good sequences, simple elements

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Abstract: The truncation operation facilitates the articulation and analysis of several aspects of the structure of archimedean vector lattices; we investigate two such aspects in this article. We refer to archimedean vector lattices equipped with a truncation as truncs. In the first part of the article we review the basic definitions, state the (pointed) Yosida representation theorem for truncs, and then prove a representation theorem which subsumes and extends the (pointfree) Madden representation theorem. The proof has the virtue of being much shorter than the one in the literature, but the real novelty of the theorem lies in the fact that the topological data dual to a given trunc G is a (localic) compactification, i.e., a dense pointed frame surjection $q\colon M \to L$ out of a compact regular pointed frame M. The representation is an amalgam of the Yosida and Madden representations; the compact frame M is sufficient to describe the behavior of the bounded part G^* of G in the sense that $\widetilde{G}^*$ separates the points of the compact Hausdorff pointed space X dual to M, while the frame L is just sufficient to capture the behavior of the unbounded part of G in $\mathcal{R}_0 L$. The truncation operation lends itself to identifying those elements of a trunc which behave like characteristic functions, and in the second part of the article we characterize in several ways those truncs composed of linear combinations of such elements. Along the way, we show that the category of such truncs is equivalent to the category of pointed Boolean spaces, and to the category of generalized Boolean algebras. The short third part contains a characterization of the kernels of truncation homomorphisms in terms of pointwise closure. In it we correct an error in the literature.

Keywords: truncated archimedean vector lattice; pointwise convergence; l-group; completely regular pointed frame
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