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Chromatic number of the product of graphs, graph homomorphisms, antichains and cofinal subsets of posets without AC

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Abstract: In set theory without the axiom of choice (AC), we observe new relations of the following statements with weak choice principles. ◦ If in a partially ordered set, all chains are finite and all antichains are countable, then the set is countable. ◦ If in a partially ordered set, all chains are finite and all antichains have size $\aleph_\alpha$, then the set has size $\aleph_\alpha$ for any regular $\aleph_\alpha$. ◦ Every partially ordered set without a maximal element has two disjoint cofinal sub sets – CS. ◦ Every partially ordered set has a cofinal well-founded subset – CWF. ◦ Dilworth's decomposition theorem for infinite partially ordered sets of finite width – DT. We also study a graph homomorphism problem and a problem due to A. Hajnal without AC. Further, we study a few statements restricted to linearly-ordered structures without AC.

Keywords: chromatic number of product of graphs; ultrafilter lemma; permutation model; Dilworth's theorem; chain; antichain; Loeb's theorem; application of Loeb's theorem

AMS Subject Classification: 05C15, 03E25, 03E35

References

[1] Banaschewski B., *Algebraic closure without choice*, Z. Math. Logik Grundlag. Math. **38** (1992), no. 4, 383–385.

[2] Cowen R. H., *Generalizing Kőnig's infinity lemma*, Notre Dame J. Formal Logic **18** (1977), no. 2, 243–247.

[3] Dilworth R. P., *A decomposition theorem for partially ordered sets*, Ann. of Math. (2) **51** (1950), 161–166.

[4] Hajnal A., *The chromatic number of the product of two $\aleph_1$-chromatic graphs can be countable*, Combinatorica **5** (1985), no. 2, 137–139.

[5] Halbeisen L., Tachtsis E., *On Ramsey choice and partial choice for infinite families of n-element sets*, Arch. Math. Logic **59** (2020), no. 5–6, 583–606.

[6] Hall M. Jr., *Distinct representatives of subsets*, Bull. Amer. Math. Soc. **54** (1948), 922–926.

[7] Howard P. E., *Binary consistent choice on pairs and a generalization of König's infinity lemma*, Fund. Math. **121** (1984), no. 1, 17–23.

[8] Howard P., Rubin J. E., *Consequences of the Axiom of Choice*, Mathematical Surveys and Monographs, 59, American Mathematical Society, Providence, 1998.

[9] Howard P., Saveliev D. I., Tachtsis E., *On the set-theoretic strength of the existence of disjoint cofinal sets in posets without maximal elements*, MLQ Math. Log. Q. **62** (2016), no. 3, 155–176.

[10] Howard P., Tachtsis E., *On vector spaces over specific fields without choice*, MLQ Math. Log. Q. **59** (2013), no. 3, 128–146.

[11] Jech T. J., *The Axiom of Choice*, Studies in Logic and the Foundations of Mathematics, 75, North-Holland Publishing Co., Amsterdam, American Elsevier Publishing Co., New York, 1973.

[12] Keremedis K., *The compactness of $2^{\mathbb{R}}$ and the axiom of choice*, MLQ Math. Log. Q. **46** (2000), no. 4, 569–571.

[13] Komjáth P., Totik V., *Problems and Theorems in Classical Set Theory*, Problem Books in Mathematics, Springer, New York, 2006.

[14] Loeb P. A., *A new proof of the Tychonoff theorem*, Amer. Math. Monthly **72** (1965), no. 7, 711–717.

[15] Łoś J., Ryll-Nardzewski C., *On the application of Tychonoff's theorem in mathematical proofs*, Fund. Math. **38** (1951), no. 1, 233–237.

[16] Tachtsis E., *On Martin's axiom and forms of choice*, MLQ Math. Log. Q. **62** (2016), no. 3, 190–203.

[17] Tachtsis E., *On Ramsey's theorem and the existence of infinite chains or infinite anti-chains in infinite posets*, J. Symb. Log. **81** (2016), no. 1, 384–394.

[18] Tachtsis E., *On the minimal cover property and certain notions of finite*, Arch. Math. Logic. **57** (2018), no. 5–6, 665–686.

[19] Tachtsis E., *Dilworth's decomposition theorem for posets in ZF*, Acta Math. Hungar. **159** (2019), no. 2, 603–617.

[20] Tachtsis E., *Łoś's theorem and the axiom of choice*, MLQ Math. Log. Q. **65** (2019), no. 3, 280–292.