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On subcompactness and countable subcompactness of metrizable spaces in ZF

Comment.Math.Univ.Carolin. 63,2 (2022) 229 –244.

Abstract: We show in ZF that: (i) Every subcompact metrizable space is completely metrizable, and every completely metrizable space is countably subcompact. (ii) A metrizable space $\mathbf{X} = (X, T)$ is countably compact if and only if it is countably subcompact relative to T. (iii) For every metrizable space $\mathbf{X} = (X, T)$, the following are equivalent: (a) $\mathbf{X}$ is compact; (b) for every open filter $\mathcal{F}$ of $\mathbf{X}$, $\bigcap\{\overline{F} : F \in \mathcal{F}\} \neq \emptyset$; (c) $\mathbf{X}$ is subcompact relative to T. We also show: (iv) The negation of each of the statements, (a) every countably subcompact metrizable space is completely metrizable, (b) every countably subcompact metrizable space is subcompact, (c) every completely metrizable space is subcompact, is relatively consistent with ZF. (v) AC if and only if for every family $\{\mathbf{X}_i : i \in I\}$ of metrizable subcompact spaces, for every family $\{\mathcal{B}_i : i \in I\}$ such that for every $i \in I$, $\mathcal{B}_i$ is a subcompact base for $\mathbf{X}_i$, the Tychonoff product $\mathbf{X} = \prod_{i \in I} \mathbf{X}_i$ is subcompact with respect to the standard base $\mathcal{B}$ of $\mathbf{X}$ generated by the family $\{\mathcal{B}_i : i \in I\}$.

Keywords: axiom of choice; compact; countably compact; subcompact; countably subcompact; lightly compact metric space
AMS Subject Classification: 03E25, 54D30, 54E35, 54E45, 54E50

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