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On the recognizability of some projective general linear groups by the prime graph

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Abstract: Let G be a finite group. The prime graph of G is a simple graph $\Gamma(G)$ whose vertex set is $\pi(G)$ and two distinct vertices p and q are joined by an edge if and only if G has an element of order pq. A group G is called k-recognizable by prime graph if there exist exactly k nonisomorphic groups H satisfying the condition $\Gamma(G) = \Gamma(H)$. A 1-recognizable group is usually called a recognizable group. In this problem, it was proved that $\mathrm{PGL}(2, p^\alpha)$ is recognizable, if p is an odd prime and $\alpha > 1$ is odd. But for even α, only the recognizability of the groups $\mathrm{PGL}(2, 5^2)$, $\mathrm{PGL}(2, 3^2)$ and $\mathrm{PGL}(2, 3^4)$ was investigated. In this paper, we put $\alpha = 2$ and we classify the finite groups G that have the same prime graph as $\Gamma(\mathrm{PGL}(2, p^2))$ for $p = 7, 11, 13$ and 17. As a result, we show that $\mathrm{PGL}(2, 7^2)$ is unrecognizable; and $\mathrm{PGL}(2, 13^2)$ and $\mathrm{PGL}(2, 17^2)$ are recognizable by prime graph.

Keywords: projective general linear group; prime graph; recognition
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