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Fréchet differentiability via partial Fréchet differentiability

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Abstract: Let $X_1, \dots, X_n$ be Banach spaces and f a real function on $X = X_1 \times \dots \times X_n$. Let A_f be the set of all points $x \in X$ at which f is partially Fréchet differentiable but is not Fréchet differentiable. Our results imply that if $X_1, \dots, X_{n-1}$ are Asplund spaces and f is continuous (respectively Lipschitz) on X, then A_f is a first category set (respectively a σ-upper porous set). We also prove that if X, Y are separable Banach spaces and $f\colon X \to Y$ is a Lipschitz mapping, then there exists a σ-upper porous set $A \subset X$ such that f is Fréchet differentiable at every point $x \in X \setminus A$ at which it is Fréchet differentiable along a closed subspace of finite codimension and Gâteaux differentiable. A number of related more general results are also proved.

Keywords: Fréchet differentiability; partial Fréchet differentiability; first category set; Asplund space; σ-porous set
AMS Subject Classification: 46G05, 46T20

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