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A note on nonseparable Lipschitz-free spaces

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Abstract: We prove that several classical Banach space properties are equivalent to separability for the class of Lipschitz-free spaces, including Corson's property $(\mathcal{C})$, Talponen's countable separation property, or being a Gâteaux differentiability space. On the other hand, we single out more general properties where this equivalence fails. In particular, the question whether the duals of nonseparable Lipschitz-free spaces have a weak* sequentially compact ball is undecidable in ZFC. Finally, we provide an example of a nonseparable dual Lipschitz-free space that fails the Radon–Nikodým property.

Keywords: Lipschitz-free space; nonseparable Banach space; sequentially compact; Radon–Nikodým property
AMS Subject Classification: 46B20, 46B26, 46E15

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