

Hooriye S. Jalali Ghamsari, Mahdi Dehghani

A generalized Birkhoff–James orthogonality and norm parallelism in unital C^ -algebras and their characterizations*

Comment.Math.Univ.Carolin. 66,1 (2025) 47–69.

Abstract: Let \mathcal{A} be a unital C^* -algebra and let $a \in \mathcal{A}$ be a positive and invertible element. Suppose that $\mathcal{S}(\mathcal{A})$ is the set of all states on \mathcal{A} and let

$$\mathcal{S}_a(\mathcal{A}) = \left\{ \frac{f}{f(a)} : f \in \mathcal{S}(\mathcal{A}), f(a) \neq 0 \right\}.$$

We introduce a family of generalized norms, called (a, λ) -norms, on \mathcal{A} defined by

$$\|x\|_{a,\lambda} := \sup \left\{ \sqrt{\lambda\varphi(x^*ax) + (1-\lambda)|\varphi(ax)|^2} : \varphi \in \mathcal{S}_a(\mathcal{A}) \right\}, \quad \lambda \in [0, 1].$$

This family of norms generalizes the recently introduced a -operator norm, $\|\cdot\|_a$ and a -numerical radius norm, $v_a(\cdot)$ in unital C^* -algebras. The notions of Birkhoff–James orthogonality and norm-parallelism with respect to $\|\cdot\|_{a,\lambda}$, which is called, (a, λ) -Birkhoff–James orthogonality and (a, λ) -norm parallelism in \mathcal{A} , respectively, are introduced and investigated. Characterizations of (a, λ) -norm parallelism and (a, λ) -Birkhoff–James orthogonality in terms of the elements of $\mathcal{S}_a(\mathcal{A})$ are obtained. In particular, the relationship between these new concepts are described. Our results extend and cover some known results in this area.

Keywords: a -numerical radius; a -Birkhoff–James orthogonality; a -norm parallelism; a -numerical radius parallelism; C^* -algebra; state space; a -numerical range

AMS Subject Classification: 46L05, 47A12, 46B20, 46C50

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