Historical notes on loop theory

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Abstract. This paper deals with the origins and early history of loop theory, summarizing the period from the 1920s through the 1960s.

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This paper is an attempt to map, to fit together not only in a geographical and a chronological sense but also conceptually, the various areas where loop theory originated and through which it moved during the early part of its 70 years of history. 70 years is not very much compared to, say, over 300 years of differential calculus. But it is precisely because loop theory is a relatively young subject that it is often misinterpreted. Therefore, it is extremely important for us to acknowledge its distinctive origins.

To give an example, when somebody asks, "What is a loop?", the simplest way to explain is to say, "it is a group without associativity". This is true, but it is not the whole truth. It is essential to emphasize that loop theory is not just a generalization of group theory but a discipline of its own, originating from and still moving within four basic research areas — algebra, geometry, topology, and combinatorics.

Looking back on the first 50 years of loop history, one can see that every decade initiated a new and important phase in its development. These distinct periods can be summarized as follows:

I.	1920s	the first glimmerings of non-associativity
II.	1930s	the defining period (Germany)
III.	1940s-60s	building the basic algebraic frame and new approaches to projective geometry (United States)
IV.	1950s-60s	the return of loops to Western Europe (England, France, Germany, Holland, and Italy)
V.	1960s	the ascendance of loops in the Soviet Union

Each of the periods II-V is represented in our literature by a classic book of that era, which to this day remains a main reference source in its respective area:

II. *Geometry of Webs*, by Blaschke and Bol, 1938 (German)

This paper is in final form and no version of it will be submitted for publication elsewhere.

III. *A Survey of Binary Systems*, by Bruck, 1958 (English)
IV. *Projective Planes*, by Pickert, 1955 (German)
V. *Foundations of the Theory of Quasigroups and Loops*, by Belousov, 1967 (Russian)

One aim of this paper is to shed light on the original motivations for the first publications on quasigroups by Moufang and Bol. The events of those years are too far in the past for many people to know first-hand or to have heard about from witnesses. But they are also too recent to be found in math-history books or even on any of the new math-historical web-sites.

Let us begin with period I, or rather with the prehistory of non-associativity.

I. 1920s — the first glimmerings of non-associativity

In the history of science we know many cases where at certain times certain revolutionary ideas were, so to speak, "in the air" until they manifested themselves in different places, sometimes independently and in different forms.

During the last two centuries, two such prominent cases have occurred in mathematics and physics. Hyperbolic geometry was discovered almost simultaneously by Lobatschevski and Bolyai in the 1820s, and would subsequently combine with Riemannian geometry (announced in 1866) to form the field of Non-Euclidean Geometry. Similarly, around the turn of the century, an entirely new conception of Space-Time emerged out of the Lorentz transformation of 1895, which replaced earlier Galilean notions, and Einstein's Special Relativity of 1905. We now know that these two ideas, Non-Euclidean Geometry and Curved Space-Time, are not utterly unrelated and both helped to prepare the ground for the notion of non-associativity.

The oldest non-associative operation used by mankind was plain subtraction of natural numbers. But the first example of an abstract non-associative system was Cayley numbers, constructed by Arthur Cayley in 1845. Later they were generalized by Dickson to what we know as Caley-Dickson algebras. They became the subject of vigorous study in the 1920s because of their prominent role in the structure theory of alternative rings.

Another class of non-associative structures was systems with one binary operation. One of the earliest publications dealing with binary systems that explicitly mentioned non-associativity was the paper *On a Generalization of the Associative Law* (1929) by Anton K. Suschkewitsch, who was a Russian professor of mathematics in Voronezh.

In his paper, Suschkewitsch observes that, in the proof of the Lagrange theorem for groups, one does not make any use of the associative law. So he rightly conjectures that it could be possible to have non-associative binary systems which satisfy the Lagrange property. He constructs two types of such so-called "general groups", satisfying his Postulate A or Postulate B. In Suschkewitsch's approach, one can detect some early attempts in the direction of modern loop theory as a generalization of group-theoretical notions. His "general groups" seem to be the predecessors of modern quasigroups as isotopes of groups.[1]

Unfortunately, Suschkevitsch's ideas did not take root in his own country at the time. In the 1930s, a ruthless period of political repression began in the Soviet Union and authorities did not consider Suschkewitsch to be politically trustworthy. He had descended from minor Russian nobility. What was even worse, he had studied mathematics in Berlin before the revolution and published his 1929 paper in an American journal. During his later teaching career in Khar'kov, Suschkewitsch was not permitted to supervise any dissertations — a circumstance that precluded any further development or dissemination of his ideas. It would not be until the late 1930s, with Murdoch in America, and in the 1960s, with Belousov in Russia, that Suschkevitsch's work would once again attract attention.

II. 1930s — the defining period

For the next stage, we must look to Germany in the 1930s, where interest was arising simultaneously from algebra, geometry, and topology. Geometry was traditionally strong in Germany from the age of Gauss, whereas the algebra of the XIX century was predominantly British. The state of German algebra, however, began to change considerably from the time of Hilbert and his axiomatic approach to algebra and geometry.

By the 1920s, along with Goettingen University, the relatively young University of Hamburg was becoming a new center of research, a circumstance that contributed importantly to the emergence of quasigroups. This research moved along several paths, which soon became interwoven.

On the algebraic scene, brilliant algebraists happened to be in Hamburg at the time, such as Erich Hecke, a student of Hilbert; Emil Artin; and Artin's students, Max Zorn and Hans Zassenhaus. Algebraic interest in non-associativity first came not from binary systems, as was the case with Suschkevitsch, but from alternative algebras. Many hoped that they could be useful in the mathematics of quantum mechanics. That hope was never realized, but in the process the significance of non-associativity began to emerge. It was around this time that Artin proved a theorem that Moufang would later use in her famous paper on quasigroups.

Artin's theorem: In an alternative algebra, if any three elements multiply associatively, they generate a subalgebra.

Simultaneously, along the geometric path, research was following Hilbert's principle that geometric axioms of planes correspond to algebraic properties of their coordinatizing systems.

Very exciting developments were also under way in differential geometry. There is no question that the most prominent figure at the Hamburg Mathematical Seminar at the time was Wilhelm Blaschke. His magnetic personality and innovative

[1] It was brought to my attention after the original presentation of this paper that binary systems with left and right division, which we now call quasigroups, were mentioned by Ernst Schroeder in his book *Lehrbuch der Arithmethik und Algebra* (1873), and in his *Vorlesungen ueber die Algebra der Logik* (1890).

work in differential geometry attracted to him and to his ideas many young and talented mathematicians. The new branch of mathematics that he was creating was Web Geometry. Blaschke's book on the subject, *Web Geometry*, co-authored with Bol, came out only in 1938, but was preceded by many separate publications on this topic by himself and by his followers, including the 66 papers of the series "Webs and Groups". Bol alone contributed 14 papers to this series. Among the earliest were papers by Thomsen and Reidemeister, whose names we now know from corresponding web-configurations. The title of the series, "Webs and Groups", subtitled "Topological Questions of Differential Geometry", was also significant for having neatly combined the three main areas from which loop theory emerged: geometry, algebra, and topology. The topological aspect entered from such questions as linearization: whether there exists a unique topological transformation that maps a web of curves into a linear web. Bol, among others, made significant contributions to this area of research.

From the point of view of loop theory, all these developments culminated in the appearance of two papers that defined the two most important classes of loops as we know them now, Moufang loops and Bol loops: *Zur Struktur von Alternativkoerpern* by Ruth Moufang (1935), and *Gewebe und Gruppen* by Gerrit Bol (1937). Together, these papers marked the formal beginning of loop theory.

Let us first look at Moufang's paper, which was motivated by a publication by Max Zorn on alternative rings in which Zorn used Artin's theorem. Moufang starts with an alternative field and endeavors to prove Artin's theorem using the multiplicative system only. She defines a structure, which she calls a Quasigroup Q^*, satisfying the following postulates:

(1), (2) closure, existence of an identity element and unique inverses
(3) $a(a'b) = (aa')b$ and $(ba')a = b(a'a)$
(4) $[a(ca)]b = a[c(ab)]$

She also defines a system Q^{**}, believing it to be different from Q^*. Q^{**} satisfies an additional identity:

(5) $(ab)(ca) = a[(bc)a]$

Bol soon showed that (4) implies (5), and Bruck later proved that they both are equivalent to two other identities:

(6) $[(ab)c]b = a[b(cb)]$

One can see that system Q^* is what is now known as a Moufang loop, which can be defined by any one of the Moufang identities (4) through (6).

Moufang proves that Q^* is diassociative — the subquasigroup generated by any two elements is associative — and satisfies a theorem that echoes Artin's theorem and is now known as Moufang's theorem. She also gives a geometric interpretation to Artin's theorem for projective planes, showing that a projective plane satisfies the Complete Quadrilateral theorem if and only if it can be coordinatized by an alternative field. Today we call such a plane a Moufang plane.

Moufang (1905-1977), along with Sofia Kovalevskaya and Emmy Noether, was one of the first three women to make their names in mathematics. It is difficult to resist a note of a feminist nature in expressing pride that our field was founded by another woman.

Moufang, a geometer, had studied at the University of Frankfurt, and later in Koenigsberg, where she was strongly influenced by Reidemeister. Before her paper on quasigroups, she published 14 papers on topics of geometry. Upon earning her Ph.D. at Koenigsberg, Moufang returned to Frankfurt with the intention of pursuing an academic career. She was soon informed by the Ministry of Education of the Third Reich, however, that as a woman, she was not permitted to teach at a predominantly male university. She would only be allowed to do research work. So, Moufang was employed from 1937 as an industrial mathematician at the Krupp Corporation. She returned to Frankfurt University after the war, and became the first female professor of mathematics in Germany.

Moufang never published again, but fortunately lived long enough to witness many of the fruits of her early work, seeing her name attached to such things as Moufang loops and Moufang planes. Today, of course, there are many more new branches of that Moufang tree — Moufang symmetries, Moufang polygons, Moufang buildings — not only in algebra and geometry, but in other fields as well, including mathematical physics.

The next most important paper on the subject of quasigroups appeared two years after Moufang's: *Gewebe und Gruppen* by Gerrit Bol (1937). Bol's approach is from a web-geometrical point of view. He constructs three new configurations U_1, U_2, U_3 and asks whether the closure of these three figures implies associativity. He answers that question in the negative, and shows that the three U figures together imply only the law $a[b(cb)] = [(ab)c]b$, which is precisely one of the Moufang identities. To demonstrate this fact, Bol gives an example constructed by Zassenhaus. This example (of order 81) was, in fact, the first example of a non-associative commutative Moufang loop.

Further, Bol explains the algebraic meaning of each of the U figures and shows that U_1 and U_2 correspond to laws that we now call the right Bol and the left Bol identities, respectively: $a((bc)b) = ((ab)c)b$ and $(b(cb))a = b(c(ba))$. It was Zassenhaus, again, who soon constructed the first example of a right Bol loop.

Bol also proves the following properties implied by the U figures:

U_1	the right inverse property:	$(dc)c' = d$
U_2	the left inverse property:	$a'(ab) = b$
U_3	the anti-automorphic inverse law:	$(ab)' = b'a'$

Bol shows that U_1 and U_2 together imply U_3, and when all three are closed, one obtains Moufang's quasigroup Q^*. Bol also demonstrates that Moufang's Quasigroup Q^* satisfies the flexible law: $b(cb) = (bc)b$.

Thus Bol practically split the Moufang identity in two, showing that, in our language, a loop is Moufang if and only if it is both right and left Bol. The irony was that Bol was not originally aware of Moufang's work when he wrote his paper.

In a footnote, he acknowledged that her paper had only come to his attention after he had practically completed his article. Here we have yet another example of a simultaneous development of similar ideas from different perspectives.

Neither Moufang nor Bol ever returned to the subject of quasigroups, although Bol would publish prolifically in his later life on questions of differential geometry. When we invited Bol to a conference on Moufang and Bol loops on the occasion of his seventieth birthday in Oberwolfach in 1976, he came and patiently listened to the lecture of D.A. Robinson, who was the first mathematician to undertake serious study of Bol loops in the 1960s. Later Bol confided to us, however, that the subject bearing his own name was of little interest to him any longer, and that it was instead archeology that had captured his imagination since retirement.

After Moufang and Bol, development of the field came to a halt in Germany due to the political situation and the onset of war. Gerrit Bol (1906-1989), being Dutch, had to join the Dutch military in 1940 and was soon taken prisoner of war by the Germans. His mentor Blaschke was able to rescue him, however, and he continued his work in Germany. From 1945 onward, he served as professor at Freiburg.

III. 1940s-60s — building the basic algebraic frame and new approaches to projective geometry

After the demise of quasigroups in Germany, it was the United States that became the new center of research on the subject. Why the United States? There were mainly two reasons.

One important factor was that many prominent mathematicians had to leave Germany when Hitler came to power, either being Jews themselves or having Jewish spouses. Among them were many who had already been exposed to the subject of quasigroups, such as Emil Artin, Reinhold Baer, Richard Brauer, and Max Zorn. Many of them came to the United States and continued their work in this field, notably Reinhold Baer.

The other important reason was that a strong interest in non-associative structures already existed in the United States, particularly at the University of Chicago. Leonard Dickson, whose name we know from Cayley-Dickson algebra, was teaching at Chicago, along with his former student Abraham Adrian Albert. In this way, Chicago became a new center of quasigroup research in the 1940s, just as Hamburg had become in the previous decade.

In addition to alternative algebra research, there were already several American publications on quasigroups:

1937 *Theory of Quasi-Groups*, by Hausmann and Ore;
1939 *Quasi-Groups Which Satisfy Certain Generalized Associative Laws*, by Murdoch;
1940 *Quasi-Groups*, by Garrison.

All three authors already used the term “quasigroup” in a broader sense, the way we use it now, and not just as Moufang's Q^* system. The first two authors still assumed the existence of an identity element, whereas Murdoch already considered the case of a one-sided identity.

It was at this point that the terminology of quasigroup theory underwent a historic change. It became apparent that it was necessary to distinguish between two classes of quasigroups: those with and those without an identity element. A new name was needed to designate the system with identity. This occurred around 1942, among people of Albert's circle in Chicago, who coined the word "loop" after the Chicago Loop. For Chicago locals, the term "Loop" designated the main business area and the elevated train that literally made a loop around this part of the city.

It was a brilliant choice in several senses. First, the word "loop" rhymes with "group". Second, it expresses a sense of closure. And third, it is short and simple, so that it could be easily adopted in other languages. Today, it is used in many languages, with slight variations: for example, DIE LOOP in German (first used by Pickert) and LUPA in Russian. The French are, of course, an original and non-conforming people, so in French it is LA BOUCLE.

The first publications introducing the term "loop" were the two very important papers that Albert wrote in 1943: *Quasigroups. I* and *Quasigroups. II.* In addition to introduction of the new term "loop", a highly significant aspect of the *Quasigroups. I* paper was the introduction of the concept of isotopy for quasigroups. Albert's papers were soon followed by two very important publications by Richard Hubert Bruck: *Some Results in the Theory of Quasigroups* (1944) and *Contributions to the Theory of Loops* (1946). Without a doubt, in this American period of loop theory, stretching from the 1940s through the 1960s, the most important role has to be ascribed to Albert and Bruck and their schools.

For the younger generation of today, the concept of "schools" may seem outmoded. In this new age of easy cyber-communications, one can reach practically anybody anywhere by e-mail and internet to ask questions and, if lucky, get answers or information. But it was different in the early days of the field, when important schools inspired and promoted the development of new ideas. A "school" was not just a school of thought, but often also associated with the strong personality of its leader and a certain geographic proximity. In those days much depended upon oral communication, which might take place through discussions around a table, during long hikes, and so forth. There were personalities like Blaschke's, schools like Albert's in Chicago, or Reinhold Baer's after his return to Frankfurt after the war. One of the most successful and prolific schools centered around Bruck in Madison, Wisconsin.

The volume of research done during this period by Albert and Bruck and their followers is so enormous that it is not possible to account for all their contributions in a short paper or to mention names without running the risk of omitting some important ones. Among the chief topics of research were the following:

Isotopy theory — properties of isotopic quasigroups and loops, isotopic invariants,

autotopisms, pseudo-automorphisms, isotopy-isomorphy property;
Loops with different inverse properties — left, right, weak, cross, automorphic, and anti-automorphic inverse properties;
Basic concepts of subquasigroups, cosets, "characteristic" property π and nilpotency with respect to π;
Homomorphy theory;
Groups of permutations on loops — multiplication groups, inner mappings and the notion of A-loops, semi-automorphisms;
Moufang loops;
Commutative Moufang loops;
Bol loops and their sub-variety of Bruck loops;
Different classes of quasigroups — totally symmetric, distributive, abelian, and through them, different geometric and combinatorial systems.

Bruck's book *A Survey of Binary Systems* appeared in 1958 and remains even today the most referred-to text on loops.

One can see that during this period, from the 1940s through the 1960s, the basic algebraic frame of loop theory was erected. Loop theory had gained a firm ground that would allow it to move in new directions and flourish in other places. We will return later to the geometric achievements of this time, but let us first follow the algebraic branch in post-war Western Europe.

IV. 1950s-60s — the return of loops to Western Europe

One of the many new directions in which loop theory began to move was toward the universal algebra approach. Universal algebra had been one of those "new ideas in the air" during the 1930s. Stemming from Emmy Noether in Goettingen, it quickly spread throughout the United States and Europe: B.H. Neumann in Cambridge, G. Birkhoff at Harvard, and A.I. Mal'cev in the Soviet Union.

In England, the extension of its concepts to quasigroups and loops became the life work of Trevor Evans from the late 1940s onward. By defining quasigroups as algebras with three operations, including the left and the right division, Evans was able to consider quasigroups as varieties of Ω-algebras, $\Omega = \{\cdot, \backslash, /\}$, and apply to them many of the notions and tools of universal algebra.

Another area of research in England at the time was different classes of quasigroups, for example, the work done by I.M.H. Etherington on entropic quasigroups, which satisfy this identity: $(ab)(cd) = (ac)(bd)$. Etherington showed that totally symmetric entropic quasigroups are naturally connected to the geometry of plane cubic curves. The subject of cubic hypersurfaces was later fully developed by Yuri Manin in the Soviet Union.

Interestingly, the classes of entropic and left-distributive quasigroups were also studied by some Japanese mathematicians, including Takasaki in Harbin, as early as the late 1930s and early 1940s, although there is not much documentation of it in western literature.

Still another area of research in England at the time was that of Latin squares. The subject of Latin squares is, of course, much older than loop theory. Mutually orthogonal Latin Squares were already studied by Euler in the XVIII century from a combinatorial point of view. However, as loop theory developed, there appeared connections between the combinatorial and several quasigroup-theoretical aspects of Latin squares. For example, combinatorial structures such as bloc designs or Steiner triple systems can be associated with algebraic varieties of Steiner quasigroups and totally symmetric loops.

To give another example, Fisher and Yates, in England during the 1930s, showed that every 6 x 6 Latin square belongs to a set of six so-called "adjugates", which we now know as "conjugate" or "inverse" quasigroup operations, or "parastrophes". The concept of parastrophes in general was introduced by A. Sade in France in the 1950s. Subsequently, in the early 1960s, Rafael Artzy introduced isostrophes as products of parastrophes and isotopes.

Anthony Donald Keedwell, from the 1960s onward, studied many other connections between Latin squares, loops, and geometry — for example, between orthogonal Latin squares and neofields, which are structures with loops as their additive systems.

But let us return to the geometric branch as it developed in the 1940s in the United States and later in Europe.

Bruck himself had a geometric background as a student of Richard Brauer, and published several papers on geometry. In fact, his mathematical genealogical tree goes back to Gauss, via Weierstrass, Frobenius, Schur, and Brauer.

But the main figures in this branch of geometry in the United States in the 1940s were Reinhold Baer and Marshall Hall. Both were working on the idea of projective planes as planes over some algebraic systems. They complemented each other's work by using different approaches. Their most important and innovating papers, respectively, were *Homogeneity of Projective Planes* (1942) and *Projective Planes* (1943).

Marshall Hall worked on the correspondence between projective planes and algebraic systems such as double loops and ternary rings and fields. Reinhold Baer, on the other hand, used Felix Klein's group-theoretical approach to consider groups of collineations of projective planes. This approach proved to be very useful for finite cases. In finite cases, collineation groups also lead to questions of combinatorics — block designs and Steiner triple systems.

Both papers would stimulate extensive research in the future. In the United States, there were several people working in this area, among them R.H. Bruck, E. Kleinfeld, D.R. Hughes, T.G. Ostrom, and P. Dembowski. But much more work in this direction was later done in Europe, where the field was dominated by German and Italian geometers, among the latter Beniamino Segre and Adriano Barlotti. Baer's return to teaching at the University of Frankfurt in 1956 contributed to the flourishing of geometry in Germany.

Around this time, another approach, von Staudt's point of view, led to the well-known Lenz-Barlotti system for classification of projective planes and their algebraic structures using groups of perspectivities.

In order to describe sharply 2-transitive permutation groups, Helmut Karzel introduced, in 1968, the concept of a neardomain as a generalization of Zassenhaus's nearfield. The non-associative additive system of a neardomain was later proved to be a Bruck loop.

In the meantime, there had appeared the extremely influential book, one of our classics, *Projective Planes* by Guenter Pickert (1955). In addition to addressing questions of projective planes over algebraic structures, Pickert introduces the topic of topological planes over ternary fields, thus reviving the geometric-topological spirit of the 1930s.

Another source of topological stimuli was Hans Freudenthal in Holland, who published on octaves, their geometry, and topological projective planes in the 1950s and 1960s. If one adds to this the input of Russia's Scornjakov and Mal'cev's ideas on topological projective planes and analytic loops at this time, one can understand why the subject of topological loops would become such a fertile ground in the future.

V. 1950s-60s — the ascendance of loops in the Soviet Union

Interestingly, the topic of non-associative structures in the Soviet Union in the 1950s and 1960s seems to have repeated the same cycle that it underwent in Germany in the 1930s: geometry — topology — differential geometry — and only later quasigroups. Nevertheless, this cycle played itself out on a much higher plane, since the basic knowledge already existed by the 1950s.

The reader will find several other articles in this volume addressing subjects of differential geometry, topological loops, and of smooth quasigroups and loops as they emerged during this period. For that reason, we will take a look only at the achievements that were made in the Soviet Union along the algebraic branch by the school of Belousov.

Valentin Danilovitsch Belousov, although Moldavian, did his post-graduate work at Moscow University under A.G. Kurosh. Originally, Kurosh himself did not think highly of the subject, but Belousov was able to persist and later to build an extremely successful school in Kishenev, in today's Moldova. For the Soviet Union and the former Soviet bloc countries, Belousov's role in the success of quasigroup and loop theory, and his 1967 book *Foundations of the Theory of Quasigroups and Loops*, can rightly be compared with the role that Bruck and his *Binary Systems* had played in the United States a decade or two earlier. Unfortunately, Belousov's excellent book is not as widely known as Bruck's in the West, since it is written in the Russian language and has never been translated.

While Bruck's emphasis was on loops, in Belousov's work, the weight shifted toward quasigroups in general. As in previous periods, new aspects and new approaches emerged in the work of this school. Among them were the following:

New approaches to quasigroups — derivative operations, special and F-quasigroups;
New properties of known quasigroups — distributive being isotopic to commutative Moufang loops, left-distributive that are isotopic to groups, medial as F-quasigroups, isotopes of totally symmetric quasigroups;
n-ary quasigroups and corresponding algebraic k-webs;
Functional equations to express general laws of quasigroups (binary as well as n-ary);
Algebraic webs and their use in questions of isotopy and parastrophy of quasigroups and loops — for example, a new algebraic interpretation of Bol's U_3 configuration;
B_3-loops, which satisfy the U_3 condition and are isostrophic to left and right Bol loops;
Generalized Moufang and Bol loops.

Belousov's work greatly contributed to the spread of loop theory across Central Europe, particularly in such countries as Hungary, Romania, and the former Czechoslovakia. At the same time, early publications in these countries show the Western influence of Blaschke, Pickert, and Bruck, as one can see, for example, in papers by J. Aczél and F. Rado. The present Czech school of Tomaš Kepka, although acknowledging their historic ties with the Moldova school, have also been following many Western traditions, and are presently involved in several joint research projects with American loop theorists.

And so, in an historic sense, Prague seems an excellent choice for this first international conference on loops. Just as the ideas of many of the schools mentioned above have found a common ground here in the Czech Republic, let us hope that we too can follow their example, continue in this spirit of cooperation, and look forward to new and even more exciting periods of loop history in the new millennium.

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