On Moufang A-loops

J.D. Phillips

Abstract. In a series of papers from the 1940's and 1950's, R.H. Bruck and L.J. Paige developed a provocative line of research detailing the similarities between two important classes of loops: the diassociative A-loops and the Moufang loops ([1]). Though they did not publish any classification theorems, in 1958, Bruck's colleague, J.M. Osborn, managed to show that diassociative, commutative A-loops are Moufang ([5]). In [2] we relaunched this now over 50 year old program by examining conditions under which general — not necessarily commutative — diassociative A-loops are, in fact, Moufang. Here, we finish part of the program by characterizing Moufang A-loops. We also investigate simple diassociative A-loops as well as a class of centrally nilpotent diassociative A-loops. These results, *in toto*, reveal the distinguished positions two familiar classes of diassociative A-loops — namely groups and commutative Moufang loops–play in the general theory.

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1. Basic notions

A *loop* is a set with a single binary operation, denoted by juxtaposition, such in $xy = z$, knowledge of any two of x, y, and z specifies the third uniquely, and with a unique two-sided identity element, denoted by 1. A *diassociative loop* is a loop in which the subloop generated by any pair of elements is a group. A *Moufang loop* is a loop satisfying the identity $((xy)x)z = x(y(xz))$. Moufang loops are diassociative ([4]).

The multiplication group, $\mathrm{Mlt}(L)$, of a loop L is the subgroup of the group of all bijections on L generated by right and left translations. That is, $\mathrm{Mlt}(L) := \langle R(x), L(x) : x \in L\rangle$, where $R(x)$ (respectively, $L(x)$) is right (respectively, left) translation by x. Clearly, $\mathrm{Mlt}(L)$ acts as a permutation group on L. The subgroup of $\mathrm{Mlt}(L)$ which fixes the identity element in L is called the *inner mapping group.* An *A-loop* is a loop L for which every inner mapping is an automorphism of L. There are A-loops that are not diassociative, hence not Moufang ([1]). Thus, the focus of the Bruck-Paige program, and our focus here, is on diassociative A-loops. The class of diassociative A-loops is a proper subvariety of the variety of all loops ([1]). Two familiar subvarieties of the variety of diassociative A-loops are the variety of all groups and the variety of all commutative Moufang

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loops ([5]). The results in this paper underscore the central role assumed by these two subvarieties.

Let L be either a Moufang loop or a diassociative A-loop. The *nucleus*, $\mathrm{Nuc}(L)$, of L is the normal subloop of all elements that associate with all pairs of elements from L. That is, $\mathrm{Nuc}(L) := \{x \in L : \forall\, y, z \in L, (xy)z = x(yz)\}$. The *Moufang center*, $C(L)$, of L is the subloop of those elements that commute with every element in L. That is, $C(L) := \{x \in L : \forall\, y \in L, xy = yx\}$. The Moufang center of an A-loop is normal, while the Moufang center of a Moufang loop is not necessarily normal. The *center*, $Z(L)$, of L is the normal subloop of those nucleus elements that commute with each element in L. That is, $Z(L) = \mathrm{Nuc}(L) \cap C(L)$. Finally, we remind the reader of the standard notation for the inner mapping $T(x) := L(x^{-1})R(x)$.

2. Simple diassociative A-loops

Identifying the simple algebras of a given variety is a fundamentally important part of any serious investigation of that variety. We will see that many of the simple diassociative A-loops have a surprisingly "simple" and familiar structure. Toward that end, we recall a useful technical result.

Theorem 1. *Let L be a diassociative A-loop.*

1. *There is a homomorphism f from L to a group G given by $f(x) = K^*T(x)$, where K^* is a certain normal subgroup of the inner mapping group.*
2. *If L is Moufang, then $K^* = 1$, and hence $T(x)T(y) = T(xy)$ for each $x, y \in L$, $\ker(f) = C(L)$, and $L/C(L)$ is a group.*

Proof: [1, Theorem 3.4]. □

Corollary 2. *If L is a finite, Moufang A-loop, and if $C(L)$ is 2-divisible, then $\mathrm{Nuc}(L)$ contains all those elements in L whose orders are coprime with $|C(L)|$ (in addition to all cubes and commutators, as guaranteed by Theorem 5 below).*

Proof: Given $x, y \in L$, let $h = R(x)R(y)R(xy)^{-1}$. Since $L/C(L)$ is a group, given $z \in L$, we must have $zh = zc$ for some $c \in C(L)$. Since h is an automorphism, $|z| = |zh| = |zc|$. Thus, since $c \in C(L)$, $|c|$ divides $|z|$. So if $|z|$ is coprime with $|C(L)|$, then since $C(L)$ satisfies the Lagrange property ([3, Theorem 2]), c must be trivial and $zh = z$, and hence $z \in \mathrm{Nuc}(L)$. □

For the balance of this paper, $\ker(f)$ will refer to the kernel of the homomorphism f given in Theorem 1. For an arbitrary diassociative A-loop L, clearly $C(L) \leq \ker(f)$. If L is Moufang, Theorem 1 guarantees that $\ker(f) \leq C(L)$. We are interested in generalizing this condition. For p a prime, let $C(L_p) = \{x \in L : \forall\, y \in L, xy^p = y^px\}$. That is, the set $C(L_p)$ consists of all those elements of L that commute with all pth powers. Since clearly $C(L)$ is contained in $C(L_p)$, we generalize the setting of Theorem 1 by investigating diassociative A-loops for which $\ker(f)$ is contained in $C(L_p)$.

Theorem 3. *If L is a simple diassociative A-loop with* $\ker(f)$ *contained in* $C(L_p)$*, then either L has exponent p or L is, in fact, a group.*

PROOF: Since L is simple, $\ker(f)$ is either trivial or all of L. If $\ker(f)$ is trivial, then by Theorem 1, L is a group. Otherwise, $L = \ker(f)$ is contained in $C(L_p)$. That is, for each $x \in L$, we have $x^p \in C(L)$. Thus, L^p, the subloop generated by the pth powers of elements in L, is contained in $C(L)$. Since L is an A-loop, L^p is normal in L. Thus, L^p is either trivial or all of L. If L^p is trivial, L has exponent p. Otherwise $L^p = L \leq C(L)$, i.e., L is commutative, and hence by Osborn's result, Moufang. And of course, simple commutative Moufang loops are groups. □

Corollary 4. *If L is a simple diassociative A-loop with* $\ker(f)$ *contained in* $C(L_2)$*, then L is, in fact, a group.*

PROOF: Continuing from above, if L^2 is trivial, then L is commutative (since $abab = 1$, and this implies that $ba = a^{-1}b^{-1} = ab$) and as above, a group. □

Note: Compare Corollary 4 with [2, Theorem 7].

3. Moufang A-loops

We recall two necessary conditions for a diassociative A-loop to be Moufang:

Theorem 5. *If L is a Moufang A-loop, then*

1. *$L/\operatorname{Nuc}(L)$ is a commutative Moufang loop of exponent three, and*
2. *T is a homomorphism, i.e., $T(x)T(y) = T(xy)$.*

PROOF: 1. [2, Theorem 5].
2. Theorem 1 (2) above. □

We adopt the notation of Bruck and Paige, and let $U(x,y) := R(x)R(y)R(x)R(xyx)^{-1}$. Clearly a diassociative A-loop is Moufang if $U(x,y) = 1$ for all x and y. Bruck and Paige ([1, 3.62]) managed to establish the following useful identity involving $U(x,y)$:

$$T(x)T(y)T(x) = U(x,y)^2 T(xyx). \tag{3.1}$$

While they were able to exploit this identity in proving only one theorem ([1, Theorem 3.7]), we now use (3.1) both in the proof of the sufficiency of the two conditions in Theorem 5, as well as in generalizing Bruck's and Paige's above-mentioned result ([1, Theorem 3.7]).

Theorem 6. *If L is a diassociative A-loop for which both*

1. *$L/\operatorname{Nuc}(L)$ is a commutative Moufang loop of exponent three, and*
2. *T is a homomorphism,*

then L is Moufang.

PROOF: To simplify notation in the proof, we adopt the shorthand notation $U = U(x,y)$. Since T is a homomorphism, $L/C(L)$ is a group, hence Moufang. Thus, since $L/\operatorname{Nuc}(L)$ is also Moufang, given $z \in L$, we must have $zU = zc$ for some c in both $\operatorname{Nuc}(L)$ and $C(L)$. Since all cubes are nuclear, we have $z^3 = z^3U = (zU)^3 = (zc)^3 = z^3c^3$. So $c^3 = 1$. Notice that $zU^3 = (zc)U^2 = (zc^2)U = zc^3 = z$, and so $U^3 = 1$. But since T is a homomorphism, by (3.1) we have $U^2 = 1$. And thus $U = 1$ and L is Moufang. □

Clearly Theorems 5 and 6 combine to characterize Moufang A-loops:

Theorem 7. *A diassociative A-loop L is Moufang if and only if both*

1. *$L/\operatorname{Nuc}(L)$ is a commutative Moufang loop of exponent three, and*
2. *T is a homomorphism.*

If we weaken the requirement that T is a homomorphism, and balance this by adding a condition introduced in §2, we obtain a second characterization of Moufang A-loops.

Theorem 8. *A diassociative A-loop L is Moufang if and only if*

1. *$L/\operatorname{Nuc}(L)$ is a commutative Moufang loop of exponent three,*
2. *T is a semihomomorphism, i.e., $T(x)T(y)T(x) = T(xyx)$, and*
3. *$\ker(f)$ is contained in $C(L_2)$.*

PROOF: Necessity follows from Theorem 5. For sufficiency note that since both $L/\operatorname{Nuc}(L)$ and $L/\ker(f)$ are Moufang, given $z \in L$, we must have $zU = zn$ for some n in both $\operatorname{Nuc}(L)$ and $\ker(f)$. Since T is a semihomomorphism, by (3.1) we have $U^2 = 1$, and thus $z = zU^2 = (zn)U = zn^2$ and $n^2 = 1$. Moreover, since all cubes are nuclear, we have $z^3 = z^3U = (zU)^3 = znznzn$. Of course, this implies $z^2 = nznzn$. Since $\ker(f)$ is contained in $C(L_2)$, and since $n^{-1} = n$, we have $z^2n = nz^2 = znzn$. This in turn implies $z = nz$. So $n = 1$. And thus $U = 1$ and L is Moufang. □

4. Central nilpotence

In this section we offer a generalization of Bruck's and Paige's theorem about centrally nilpotent diassociative A-loops ([1, Theorem 3.7]), the only other theorem on centrally nilpotent diassociative A-loops in the literature. First, a preparatory lemma.

Lemma 9. *If L is a 2-divisible, diassociative A-loop such that both T is a semihomomorphism and $L/Z(L)$ is Moufang, then L is Moufang.*

PROOF: Given $z \in L$, and with the shorthand notation U, we have $zU = zc$ for some $c \in Z(L)$. Thus $z = zU^2 = (zc)U = zc^2$, and hence $c^2 = 1$. Finally, since L is 2-divisible, $c = 1$ and $U = 1$. □

Theorem 10. *If L is a centrally nilpotent 2-divisible diassociative A-loop, and if T is a semihomomorphism, then L is Moufang.*

Proof: We proceed by induction on n, the nilpotence class of L. If $n = 1$, then L is an abelian group. Assume $n \geq 2$. Then $L/Z(L)$ is a centrally nilpotent 2-divisible diassociative A-loop of nilpotency class $n - 1$. By induction, $L/Z(L)$ is Moufang. By Lemma 9, L is Moufang. □

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Department of Mathematics, Saint Mary's College of California, Moraga, CA 94575, USA

E-mail: phillips@stmarys-ca.edu

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