A class of quasigroups solving a problem of ergodic theory

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Abstract. A pointed quasigroup is said to be semicentral if it is principally isotopic to a group via a permutation on one side and a group automorphism on the other. Convex combinations of permutation matrices given by the one-sided multiplications in a semicentral quasigroup then yield doubly stochastic transition matrices of finite Markov chains in which the entropic behaviour at any time is independent of the initial state.

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1. Introduction

In an ergodic finite stationary Markov chain, the ultimate distribution of states is independent of the choice of initial state. Motivated by Robert Shaw's work on the dripping faucet as a chaotic system, the Japanese information theorist Yasuichi Horibe initiated investigation of ergodic finite Markov chains with doubly stochastic transition matrix such that the full entropic behaviour of the chain at any time is independent of the choice of initial state [Ho]. It is convenient to refer to such Markov chains as "superergodic". Horibe showed that convex combinations of the permutation matrices given by the left multiplications in a group under multiplication or a cyclic additive group under (nonassociative) subtraction yield superergodic transition matrices. He then posed the problem of identifying a class of quasigroups, including the class of groups, such that each quasigroup in the class yields a superergodic transition matrix. The purpose of the current paper is to propose a solution to Horibe's problem. The class of quasigroups proposed, namely the left semicentral quasigroups, have independent algebraic interest. They are intermediate between general group isotopes and the central quasigroups in which the diagonal forms a normal subquasigroup of the direct square. Left semicentral quasigroups (of unrestricted cardinality) form a variety of pointed quasigroups.

Horibe's work is reprised in Section 2. For further details on Markov chains, one may consult [Fe]. For further details on entropy, one may consult [As]. Section 3 introduces the class of left semicentral pointed quasigroups (Definition 3.1), and shows that they form a variety (Theorem 3.3). Section 4 then shows that finite left semicentral pointed quasigroups offer a solution to Horibe's problem

(Theorem 4.2). For further details on quasigroups and universal algebra, one may consult [SR].

2. Markov chains and Latin squares

Consider a finite, stationary Markov chain with stochastic transition matrix

$$\Pi = [\pi_{ij}]_{r\times r}, \tag{2.1}$$

so that each entry is non-negative, and $\sum_{j=1}^{r} \pi_{ij} = 1$ for $1 \le i \le r$. Use η to denote the r-dimensional column vector, each of whose entries is 1. Since $\Pi\eta = \eta$, one has $\det\,(\Pi - 1) = 0$, so there is an r-dimensional row vector π with $\pi\Pi = \pi$. The Markov chain is said to be *ergodic* if all the entries of some positive power Π^n of Π are positive. In this case, Markov's Theorem [Ho, Theorem 2] shows that there is a probability distribution row vector π satisfying $\pi\Pi = \pi$, such that each row of Π^n tends to π as n tends to infinity. In other words, the ultimate distribution of states of the chain is π, independent of the initial distribution.

For a probability row vector $\xi = [\xi_1\ \xi_2 \dots \xi_r]$, the *entropy*

$$H(\xi) = -\sum_{i=1}^{r} \xi_i \log \xi_i \tag{2.2}$$

is the expected value of the logarithm of the odds of occupying a given state. Set

$$\delta^i = [0 \dots 1 \dots 0], \tag{2.3}$$

the row vector with $\delta^i_j =$ **if** $i = j$ **then** 1 **else** 0, to be the distribution corresponding to certain occupancy of the i-th state. If the chain starts in the i-th state, the distribution after n steps is $\delta^i\Pi^n$. Recall that the stochastic transition matrix (2.1) is said to be *doubly stochastic* if $\sum_{i=1}^{r} \pi_{ij} = 1$ for $1 \le j \le r$. The doubly stochastic $r \times r$ matrices form the $(r-1)^2$-dimensional convex hull of the set of all $r!$ permutation matrices of size $r \times r$.

Definition 2.1 (cf. [Ho,(*)]). A stationary, ergodic Markov chain with doubly stochastic transition matrix (2.1) is said to be *superergodic* if

$$\forall n \ge 0,\ \forall i \ne j,\ H(\delta^i\Pi^n) = H(\delta^j\Pi^n). \tag{2.4}$$

In other words, the entropy of the state distribution after any number n of steps, starting from a single initial state, is independent of the choice of that initial state. □

Let $(Q, \cdot)$ be a quasigroup on the set $\{1, 2, \dots, r\}$. For each element q of Q, the left multiplication

$$L(q) : Q \to Q;\ x \mapsto q \cdot x \tag{2.5}$$

is a permutation of Q. Identify permutations of Q with the corresponding permutation matrices. Suppose that

$$p = [p_1\ p_2 \ldots p_r] \tag{2.6}$$

is a probability distribution vector. Consider the doubly-stochastic matrix

$$S(p,Q) = \sum_{i=1}^{r} p_i L(i). \tag{2.7}$$

Then each row of $S(p,Q)$ is a probability distribution whose entropy is $H(p)$. Note that if the entries of p are distinct, then $S(p,Q)$ is a Latin square.

Definition 2.2. A quasigroup Q of finite order r is said to satisfy *Horibe's condition* if for each probability distribution p and positive power n, there is a probability distribution p' and quasigroup Q' such that

$$S(p,Q)^n = S(p',Q'). \tag{2.8}$$

□

Horibe [Ho] observed that if a quasigroup Q satisfies the condition of Definition 2.2 (and if the probability distribution (2.6) has all entries positive), then $S(p,Q)$ is the transition matrix of a superergodic Markov chain. He showed that a group Q satisfies the condition of Definition 2.2, and effectively presented cyclic groups under subtraction as examples of non-associative quasigroups satisfying the condition. Finally, he raised the following

Problem 2.3. Identify classes of quasigroups, including the class of (finite) groups, satisfying the condition of Definition 2.2. □

It is convenient to refer to Problem 2.3 as *Horibe's Problem.*

3. Central and semicentral quasigroups

Construe quasigroups Q as algebras $(Q,\cdot,/,\backslash)$ of type $\{\cdot,/,\backslash\}\times\{2\}$, with binary operations of multiplication $\cdot$ (also denoted by juxtaposition), right division $/$, and left division $\backslash$, satisfying the identities

$$\begin{aligned} (x\cdot y)/y = x,\ \ x = y\backslash(y\cdot x)\\ (x/y)\cdot y = x,\ \ x = y\cdot(y\backslash x). \end{aligned} \tag{3.1}$$

A *pointed quasigroup* $(Q,e,\cdot,/,\backslash)$ is an algebra of type $\{(e,0)\}\cup[\{\cdot,/,\backslash\}\times\{2\}]$ such that $(Q,\cdot,/,\backslash)$ is a quasigroup with a nullary operation selecting an element e of Q. A pointed quasigroup $(Q,e,\cdot,/,\backslash)$ is a *group isotope* if there is a group $(Q,+)$ with identity e and a pair R,L of permutations of Q such that

$$x\cdot y = x^R + y^L \tag{3.2}$$

for x, y in Q (cf. [JK], [SR, §I.4]). A pointed quasigroup Q is *central* if there is an abelian group $(Q, +)$ with identity e and a pair of ρ, λ of automorphisms of $(Q, +)$ such that

$$x \cdot y = x^\rho + y^\lambda \tag{3.3}$$

for x, y in Q (cf. [NK], [JK], [Sm, 418], [CP, Theorem III.5.2]).

The solution to Horibe's Problem presented in the next section involves a class of pointed quasigroups intermediate between group isotopes and central quasigroups.

Definition 3.1 (cf. [B1], [B2]). A pointed quasigroup $(Q, e, \cdot, /, \backslash)$ is said to be (*left*) *semicentral* if there is a group $(Q, +)$ with identity e, a permutation R of Q, and an automorphism λ of $(Q, +)$, such that

$$x \cdot y = x^R + y^\lambda \tag{3.4}$$

for x, y in Q. □

The use of left multiplications in (2.7) implies the significance of left semicentral quasigroups in the solution of Horibe's Problem. Dually, however, one could define the class of right semicentral quasigroups using $x \cdot y = x^\rho + y^L$ with an automorphism ρ and permutation L, in place of (3.4).

In the group isotope $(Q, +)$ of a left semicentral quasigroup $(Q, e, \cdot, /, \backslash)$, denote left multiplications by

$$L_+(q) : Q \to Q;\ x \mapsto q + x \tag{3.5}$$

for q in Q.

Lemma 3.2. *For an automorphism θ of $(Q, +)$, and for q in Q, one has*

$$\theta^{-1} L_+(q)\theta = L_+(q\theta). \tag{3.6}$$

Proof: For x in Q, one has $x\theta^{-1}L_+(q)\theta = (q + x^{\theta^{-1}})\theta = q^\theta + x = xL_+(q\theta)$. □

Theorem 3.3. *The class of left semicentral pointed quasigroups $(Q, e, \cdot, /, \backslash)$ forms a variety of algebras of type $\{(e, 0)\} \cup [\{\cdot, /, \backslash\} \times \{2\}]$.*

Proof: By (3.4), one has $xR = x \cdot e$ and

$$x + y = x^{R^{-1}} \cdot y^{\lambda^{-1}} \tag{3.7}$$

for x, y in Q. Moreover, $y\lambda = e + y^\lambda = e^{R^{-1}} y = (e/e)y$. Thus

$$x + y = (x/e) \cdot ((e/e) \backslash y) \tag{3.8}$$

(cf. [SR, I(4.4)]). The associativity of the quasigroup multiplication (3.8), and the fact that the permutation $\lambda : y \mapsto (e/e) \cdot y$ is an automorphism of (3.8), are then expressible as identities in the language of pointed quasigroups. □

Remark 3.4. Belyavskaja and Tabarov [B1, Corollary 2] have shown that unpointed left semicentral quasigroups ("left linear" quasigroups in their notation) are characterized by the identity

$$[x(u \setminus y)]z = [x(u \setminus u)] \cdot (u \setminus yz). \tag{3.9}$$

Of course, one should add nonemptiness to this characterization. □

4. Semicentrality and superergodicity

In this section, it will be shown that the class of finite, semicentral quasigroups offers a solution to Horibe's Problem.

Proposition 4.1. *Let $(Q, e, \cdot, /, \backslash)$ be a left semicentral quasigroup of order r, principally isotopic to the group $(Q, +, e)$ via a permutation R and automorphism λ. Let α and β be automorphisms of $(Q, +)$. Let p and p' be r-dimensional probability row vectors as in* (2.6). *Then*

$$\left[\sum_{i=1}^{r} p_i \alpha L(i)\right] \left[\sum_{j=1}^{r} p'_j \beta L(j)\right] = \sum_{i=1}^{r} \sum_{j=1}^{r} p_i p'_j \alpha\lambda\beta L(j i^{R\beta}). \tag{4.1}$$

Proof: Using Lemma 3.2 and (3.4), one has

$$\begin{aligned} \alpha L(x) \beta L(y) &= \alpha\lambda L_+(x^R) \beta\lambda L_+(y^R) \\ &= \alpha\lambda\beta\lambda\lambda^{-1}\beta^{-1} L_+(x^R) \beta\lambda L_+(y^R) \\ &= \alpha\lambda\beta\lambda L_+(x^{R\beta\lambda}) L_+(y^R) \\ &= \alpha\lambda\beta\lambda L_+(y^R + x^{R\beta\lambda}) \\ &= \alpha\lambda\beta L(y x^{R\beta}) \end{aligned}$$

for x, y in Q. □

Theorem 4.2. *Each finite, left semicentral pointed quasigroup satisfies Horibe's condition.*

Proof: Let $(Q, e, \cdot, /, \backslash)$ be a left semicentral quasigroup on $Q = \{1, 2, \ldots, r\}$, principally isotopic to the group $(Q, +, e)$ via a permutation R and automorphism λ. Let α and β be automorphisms of $(Q, +)$. Let Q_1 be the left semicentral quasigroup on Q with multiplication $x^R + y^{\alpha\lambda}$. Let Q_2 be the left semicentral quasigroup on Q with multiplication $x^R + y^{\beta\lambda}$. Let Q_3 be the left semicentral quasigroup on Q with multiplication $x^R + y^{\alpha\lambda\beta}$. Then (4.1) shows that, for probability distributions p^1 and p^2 on Q, there is a probability distribution p^3 on Q such that

$$S(p^1, Q_1) S(p^2, Q_2) = S(p^3, Q_3). \tag{4.2}$$

Using (4.2), Horibe's condition (2.8) follows by induction on n. □

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