Moscow spaces, Pestov-Tkačenko Problem, and C-embeddings

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Abstract. We show that there exists an Abelian topological group G such that the operations in G cannot be extended to the Dieudonné completion μG of the space G in such a way that G becomes a topological subgroup of the topological group μG. This provides a complete answer to a question of V.G. Pestov and M.G. Tkačenko, dating back to 1985. We also identify new large classes of topological groups for which such an extension is possible. The technique developed also allows to find many new solutions to the equation $\upsilon X \times \upsilon Y = \upsilon(X \times Y)$. The key role in the approach belongs to the notion of Moscow space which turns out to be very useful in the theory of C-embeddings and interacts especially well with homogeneity.

Keywords: Moscow space, Dieudonné completion, Hewitt-Nachbin completion, C-embedding, G_δ-dense set, topological group, Souslin number, tightness, canonical open set

Classification: Primary 54D50, 54D60; Secondary 54C35

§0. Introduction

In 1985 V.G. Pestov and M.G. Tkačenko asked the next question [15], [20]:

Problem 0.1. Let G be a topological group, and μG the Dieudonné completion of the space G. Can the operations in G be extended to μG in such a way that μG becomes a topological group containing G as a topological subgroup?

Recall that the Dieudonné completion μG of G is the completion of G with respect to the maximal uniformity on G compatible with the topology of G. It is well known that the Dieudonné completion of a topological space X is always contained in the Hewitt-Nachbin completion υX of X (see [9], [13]). In fact, μX is the smallest Dieudonné complete subspace of υX containing X (this is a part of the folklore; see below the proof of Proposition 1.4 for details.) Moreover, if there are no Ulam-measurable cardinals, then υX and μX coincide (see [9]). Therefore, the next question, also belonging to Pestov and Tkačenko, is almost equivalent to Problem 0.1:

Problem 0.2. Let G be a topological group, and υG the Hewitt-Nachbin completion of the space G. Can the operations on G be extended to υG in such a way that υG becomes a topological group containing G as a topological subgroup?

Clearly, if there exists an Ulam-measurable cardinal τ, then for any discrete group G of cardinality τ the answer to Problem 0.2 is in negative (since in this case the Hewitt-Nachbin completion υG is a non-discrete non-homogeneous space).

Even a consistent counterexample to Problem 0.1 was not known, though large classes of topological groups were found for which the answer to Problem 0.1 is in positive (see [21], [22], [2]).

Below we answer Problem 0.1 in the negative. However, the technique developed for that allows to identify new amazingly large classes of topological groups for which the answer to Problem 0.1 is positive. Incidentally, our approach also brings to the light a rather astonishing fact that the equality $\upsilon G \times \upsilon H = \upsilon(G \times H)$ holds for many classes of topological groups. This extends results of I. Glicksberg [12], W.W. Comfort [6], W.W. Comfort and S. Negrepontis [7], M. Hušek [14] (see also [5], [10]).

All spaces considered are assumed to be Tychonoff. Notation and terminology are as in [9]. In particular, a space X is called *homogeneous* if for any two points x and y of X there exists a homeomorphism h of X onto itself such that $h(x) = y$. If A is a subset of a space X then *G_δ-closure* of A in X is defined as the set of all points $x \in X$ such that every G_δ-subset of X containing x intersects A. If X is the G_δ-closure of A, we say that A is *G_δ-dense in X*. If the G_δ-closure of A coincides with A, we say that A is *G_δ-closed*.

§1. Moscow spaces, C-embeddings, and Rajkov completion

The following notion was introduced in [1]. A space X is called *a Moscow space*, if, for each open subset U of X, the closure of U in X is the union of a family of G_δ-subsets of X.

Clearly, the notion of a Moscow space generalizes the notion of a perfectly κ-normal space introduced independently by R. Blair [4], E.V. Stchepin (see [17]), and T. Terada [18] under different names. A space X is called *perfectly κ-normal* if the closure of any open set (that is, every canonical closed set) is a zero-set. The class of Moscow spaces is much wider than the class of perfectly κ-normal spaces, since every first countable space, and even every space of countable pseudocharacter is a Moscow space while not every first countable compact space is perfectly κ-normal.

The notion of a Moscow space plays a crucial role in the theory of C-embeddings. Recall that a subspace Y of a space X is said to be *C-embedded in X*, if every continuous real-valued function f on Y can be extended to a continuous real-valued function on X. It is well known ([11]) that if a dense subspace Y of a space X is C-embedded in X, then Y is G_δ-dense in X. The converse statement is easily seen to be not true.

Obviously, *every dense subspace of a Moscow space is a Moscow space*. Here is one of our key results on Moscow spaces.

Theorem 1.1. *If a Moscow space Y is a G_δ-dense subspace of a homogeneous space X, then X is also a Moscow space and Y is C-embedded in X.*

PROOF: Let U be an open subset of X and x a point in the closure of U. Since X is homogeneous, we may assume that $x \in Y$. Then $x \in \overline{U \cap Y}$ and, since Y is a Moscow space, there exists a G_δ-set Q in the space Y such that $x \in Q \subset \overline{U \cap Y}$. Thus, there exists a countable family $\{U_n : n \in \omega\}$ of open subsets of X such that their intersection $P = \cap\{U_n : n \in \omega\}$ satisfies the condition $x \in P \cap Y \subset \bar{U}$.

We claim that $P \subset \bar{U}$. Indeed, assume the contrary. Then $P \setminus \bar{U}$ is a non-empty G_δ-subset in X, and since Y is G_δ-dense in X, it follows that $(P \setminus \bar{U}) \cap Y$ is not empty. On the other hand, $(P \setminus \bar{U}) \cap Y = (P \cap Y) \setminus \bar{U} = \emptyset$. This contradiction shows that $x \in P \subset \bar{U}$. Thus, X is a Moscow space. It remains to add that every G_δ-dense subset Y of a Moscow space X is C-embedded in X ([22], see also [19]). □

The theorem does not remain true if we drop the assumption that X is homogeneous. To see this, take Y to be an uncountable discrete space and let X be its Alexandroff one-point compactification.

Recall that *the Rajkov completion* ρG of a topological group G is the completion of G with respect to the natural two-sided uniformity of the topological group G. It is well known that ρG can be interpreted as a Rajkov complete topological group, containing G as a dense subgroup ([16]). The G_δ-closure of G in ρG will be denoted by $\rho_\omega G$. Observe that the G_δ-closure of a subgroup in a topological group H is again a subgroup of H.

The next theorem is one of our main results.

Theorem 1.2. *Let G be a Moscow topological group. Then the operations on G can be extended to the Dieudonné completion μG of G in such a way that μG becomes a topological group containing G as a topological subgroup.*

To prove Theorem 1.2, we need two preliminary results.

Proposition 1.3. *Let G be a Moscow topological group. Then $\rho_\omega G$ is a Dieudonné complete Moscow topological group in which G is C-embedded.*

PROOF: Every G_δ-closed subspace of a Dieudonné complete space is a Dieudonné complete space ([9]). Of course, $\rho_\omega G$ is G_δ-closed in ρG. It remains to refer to Theorem 1.1. □

Now we need a standard piece of technique (see [22], [20], [2]).

Proposition 1.4. *Let Z be a Dieudonné complete topological group and G a dense subgroup of Z, C-embedded in Z. Then there exists a subgroup M of Z such that $G \subset M$ and the space M is the Dieudonné completion μG of G.*

PROOF: Let M be the smallest Dieudonné complete subspace of Z such that $G \subset M$ (such subspace M exists since the intersection of any family of Dieudonné complete subspaces of Z is a Dieudonné complete space [9]). Since G is C-embedded in M, it follows that M is the Dieudonné completion of G.

It remains to show that M is a subgroup of Z. First, $M \subset M^{-1}$, since $G \subset M^{-1} \subset Z$ and M^{-1} is homeomorphic to M and, therefore, Dieudonné complete. It follows that $M^{-1} \subset (M^{-1})^{-1} = M$. Hence, $M = M^{-1}$.

For every $a \in G$ we have: $G \subset aG \subset aM \subset aZ = Z$ which implies that $M \subset aM$. Hence, $a^{-1}M \subset M$. Since $G = G^{-1}$, it follows that $aM \subset M$. Now take any $b \in M$. Then $G \subset Mb$. Indeed, take any $a \in G$. Then $ab^{-1} \in M$, that is, $ab^{-1} = c$, for some $c \in M$. It follows that $a = cb \in Mb$. Hence, $G \subset Mb$. Obviously, $Mb \subset Z$ and Mb is homeomorphic to M. Therefore, Mb is Dieudonné complete, which implies that $M \subset Mb$. Hence, $Mb^{-1} \subset M$. Since $M = M^{-1}$, it follows that $Mb \subset M$, for each $b \in M$. Now it is clear that M is closed under multiplication. Hence, M is a subgroup of Z. $\square$

PROOF OF THEOREM 1.2: It is enough to refer to Propositions 1.3 and 1.4. $\square$

Let us call a topological group G *a PT-group*, if the operations on G can be extended to the Dieudonné completion μG in such a way that G becomes a topological subgroup of μG. Now Pestov-Tkačenko Problem can be reformulated as follows: *is every topological group a PT-group*? According to Theorem 1.2, *every Moscow group is a PT-group.*

Since every space is C-embedded in its Dieudonné completion ([9], [11]), Proposition 1.4 can be reformulated as follows:

Theorem 1.5. *A topological group G is a PT-group if and only if it is C-embedded in some Dieudonné complete topological group.*

A topological group G will be called *a strong PT-group* if it is C-embedded in $\rho_\omega G$.

Corollary 1.6. *Every strong PT-group is a PT-group.*

PROOF: By Proposition 1.3, $\rho_\omega G$ is a Dieudonné complete topological group. Now it follows from Theorem 1.5 that if G is a strong PT-group, then G is a PT-group. $\square$

The converse is not true ([3]). We can reformulate Proposition 1.3 and Theorem 1.2 as follows:

Theorem 1.7. *Every Moscow topological group is a strong PT-group.*

Obviously, *every Rajkov complete topological group is also a strong PT-group while it need not be a Moscow group* (see Example 2.6 below and [3]).

Now we are going to define a cardinal invariant of tightness type for topological groups. Recall that *a right topological group* is a group with a topology on it such that the multiplication on the right is continuous. Recall also that *a canonical open subset* of a space X is any open set U in X such that U is the interior of its closure.

Let G be a right topological group, and $U \subset G$. A subset A of G is called *an ω-deep subset of U* if there exists a G_δ-subset P of G such that $e \in P$ and $AP \subset U$. We say that *the g-tightness $t_g(G)$* of a right topological group G is countable (and write $t_g(G) \le \omega$), if for each canonical open subset U of G and each $x \in \bar{U}$ there exists an ω-deep subset A of U such that $x \in \bar{A}$.

Note, that if G is a topological group such that the tightness $t(G)$ is countable, or the Souslin number $c(G)$ is countable, or the pseudocharacter of G is countable, or G is extremally disconnected, then the g-tightness of G is countable ([3]).

Theorem 1.8. *Every right topological group G of countable g-tightness is a Moscow space.*

PROOF: Take any open subset U of G, and any $x \in \bar{U}$. Obviously, we may assume that U is a canonical open subset of G. Since $t_g(G) \leq \omega$, there exists an ω-deep subset A of U such that $x \in \bar{A}$. Now we can fix a G_δ-subset P of G such that $e \in P$ and $AP \subset U$. Then $x \in xP \subset \overline{AP} \subset \bar{U}$, and xP is a G_δ-subset of G. Thus, G is a Moscow space. $\square$

Theorem 1.8 does not generalize to arbitrary topological spaces. It is really amazing how many topological conditions, which are innocently weak in the general case of arbitrary topological spaces, turn out to be sufficient for a topological group to be a Moscow space. We list some of these conditions in the next statement.

First, we recall a definition given by M.G. Tkačenko [19]. The *o-tightness* of a space X is countable (that is, $ot(X) \leq \omega$) if for each family γ of open sets and each $x \in \overline{\cup\gamma}$ there exists a countable subfamily η of γ such that $x \in \overline{\cup\eta}$. Clearly, if $t(X) \leq \omega$ or $c(X) \leq \omega$, then $ot(X) \leq \omega$.

Theorem 1.9. *Let G be a topological group satisfying at least one of the following conditions:* 1) $t(G) \leq \omega$; 2) $c(G) \leq \omega$; 3) $ot(G) \leq \omega$; 4) *points in G are G_δ-s;* 5) *G is κ-metrizable;* 6) *G is perfectly κ-normal;* 7) *G is extremally disconnected;* 8) *G is a subgroup of a topological group G such that G is a k-space;* 9) *G is totally bounded; 10) G is a subgroup of an almost metrizable group. Then G is a strong PT-group.*

PROOF: In cases 1), 2), 3), 4), and 7) this follows from Theorems 1.7 and 1.8. The cases 5) and 6) are taken care of by Theorem 1.7. If G satisfies 8), then G satisfies 3) [20]. In the cases 9) and 10) G is also a Moscow group [22] (see also [2]). It remains to refer to Theorem 1.7. Theorem 1.5 generalizes Uspenskij's result ([22]) who proved that if the o-tightness of a topological group G is countable, then it is a Moscow space and a PT-group. $\square$

§2. A non-PT-group and the formula $vG \times vH = v(G \times H)$

First, we need the next notion and two simple results about it, similar to a result in [5]. Let Y be a subspace of X, and $\mathcal{P}$ a class of spaces. We will call X *a minimal $\mathcal{P}$-extension* of Y if Y is dense in X, $X \in \mathcal{P}$, and every subspace T of X such that $Y \subset T$ and $T \in \mathcal{P}$ coincides with X. Thus, X is *a minimal Dieudonné extension* of Y if Y is dense in X, X is Dieudonné complete, and every Dieudonné complete subspace of X containing Y coincides with X. The next assertion is obvious.

Proposition 2.1. *The Dieudonné completion of X is a minimal Dieudonné extension of X.*

Proposition 2.2. *If X_i is a minimal Dieudonné extension of Y_i, $i = 1, \dots, k$, then $X = \Pi\{X_i : i = 1, \dots, k\}$ is a minimal Dieudonné extension of $Y = \Pi\{Y_i : i = 1, \dots, k\}$.*

Proof: We may assume that $k = 2$, that is, $X = X_1 \times X_2$ and $Y = Y_1 \times Y_2$. Clearly, X is Dieudonné complete and Y is dense in X. Let T be a Dieudonné complete space such that $Y \subset T \subset X$.

First, we show that $Y_1 \times X_2 \subset T$. Assume the contrary. Then there exists $(a, b) \in Y_1 \times X_2$ such that (a, b) is not in T. Then $F = \{x \in X_2 : (a, x) \in T\} = (\{a\} \times X_2) \cap T$ is a closed subspace of T containing Y_2 and $F \neq X_2$, since $b \in X_2 \setminus F$. Clearly, F is Dieudonné complete. This contradicts minimality of X_2. It follows that $Y_1 \times X_2 \subset T$. Now it remains to repeat the above argument with X_2 in the role of Y_1 and Y_1, X_1 in the roles of Y_2, X_2. Hence, $T = X_1 \times X_2$. □

We need another simple statement which should be by now obvious.

Proposition 2.3. *A topological group G of Ulam non-measurable cardinality is a strong PT-group if and only if $\mu G = \rho_\omega G (= \upsilon G)$.*

Theorem 2.4. *Let G_1 and G_2 be strong PT-groups of Ulam non-measurable cardinality. Then the formula $\upsilon(G_1 \times G_2) = \upsilon G_1 \times \upsilon G_2$ holds if and only if $G_1 \times G_2$ is a strong PT-group.*

Proof: *Necessity.* Since G_i is a strong PT-group, υG_i coincides with $\rho_\omega G_i$, by Proposition 2.3. Therefore, $\upsilon G_1 \times \upsilon G_2$ is the G_δ-closure of $G_1 \times G_2$ in $\rho G_1 \times \rho G_2$. Since $\upsilon(G_1 \times G_2) = \upsilon G_1 \times \upsilon G_2$ and $\rho G_1 \times \rho G_2 = \rho(G_1 \times G_2)$, it follows that $\upsilon(G_1 \times G_2) = \rho_\omega(G_1 \times G_2)$. Hence $G_1 \times G_2$ is C-embedded in $\rho_\omega(G_1 \times G_2)$, which means that $G_1 \times G_2$ is a strong PT-group.

Sufficiency. First of all, υG_1 and υG_2 are also topological groups, since G_1 and G_2 are PT-groups of Ulam non-measurable cardinality. Therefore, $G^* = \upsilon G_1 \times \upsilon G_2$ is a topological group as well. Since G_i is G_δ-dense in υG_i, it follows that $G_1 \times G_2$ is G_δ-dense in G^*. Consequently, since $G_1 \times G_2$ is a strong PT-group, $G_1 \times G_2$ is C-embedded in G^*. Since G^* is Hewitt-Nachbin complete, it follows that $\upsilon(G_1 \times G_2) = G^*$. Thus, $\upsilon(G_1 \times G_2) = \upsilon G_1 \times \upsilon G_2$. □

Corollary 2.5. *For any strong PT-group G and any compact group F such that $|G|$ and $|F|$ are Ulam non-measurable, the product $G \times F$ is a strong PT-group.*

Proof: Indeed, it was established in [7] that the formula $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds whenever X and Y are spaces of Ulam non-measurable cardinality at least one of which is compact. It remains to refer to Theorem 2.4. □

We now present an example of two strong PT-groups whose product is not a PT-group. The construction by which this is achieved is a modification of M. Hušek's construction in [14].

Example 2.6. Let X be a zero-dimensional pseudocompact non-compact topological group. Fix a covering η of X satisfying the next three conditions:

1) Every element of η is an open and closed subset of X;
2) No finite subfamily of η covers X;
3) The union of any finite subfamily of η belongs to η.

Consider the space $G = C_\eta(X)$ of all continuous functions on X with values in the discrete two-point group $D = \{0, 1\}$ endowed with the topology of uniform convergence on elements of η. Clearly, G is a topological group. It is also obvious that G is Rajkov complete.

Since the Souslin number of X is countable, there exists a countable subfamily γ of η such that $\cup\gamma$ is dense in X. For each $P \in \gamma$ the set U_P of all $f \in G$ such that $f(x) = 0$ for every $x \in P$ is open in G and contains the zero-function θ on X which is the neutral element of G. It is obvious that θ is the only element in $\cap\{U_P : P \in \gamma\}$. Therefore, θ is a G_δ-point in G. Since G is a topological group, it follows that the pseudocharacter of G is countable. Hence, G is a Moscow group.

The group X is also a Moscow group; actually it is even κ-metrizable and, hence, perfectly κ-normal (see [17]). Notice also that X is C-embedded in βX, since X is pseudocompact. It follows that $\mu X = \upsilon X = \beta X$, since $c(X) \le \omega$ ([16]).

Consider the natural evaluation mapping ψ of the product space $X \times G$ into the discrete space $D = \{0, 1\}$ which on this occasion we treat as a subspace of R. Clearly, ψ is continuous, since elements of η are open sets.

Claim: The group $X \times G$ is not C-embedded in $\beta X \times G$.

Observe that $\beta X \times G$ is also a topological group and $X \times G$ is G_δ-dense in $\beta X \times G$.

Let us check that ψ cannot be extended to a continuous real-valued function on $\beta X \times G$. Here we will use the property 2) of η. Since the closure in βX of any element of η is, obviously, open and βX is compact, it follows from 2) that the closures of elements of η in βX do not cover βX.

Therefore, we can choose $a \in \beta X \setminus X$ such that a does not belong to the closure of any element of η. Consider the point $(a, \theta) \in \beta X \times G$ and the subsets $B = \{(x, \theta) : x \in X\}$ and $C = \{(x, f_P) : P \in \eta, x \in X \setminus P\}$, where $f_P \in G$ is the characteristic function of $X \setminus P$, that is, $f_P(x) = 0$ for each $x \in P$ and $f_P(x) = 1$ for each $x \in X \setminus P$. Clearly, ψ takes the value 1 at each element of C and the value 0 at each element of B. Obviously, the point (a, θ) is in the closure of B. Therefore, if ψ could be continuously extended to (a, θ), the value of this extension at (a, θ) should be 0. On the other hand, a is not in the closure of any $P \in \eta$. Therefore, (a, θ) is in the closure of C as well, and the extended function should be 1 at (a, θ), a contradiction.

Finally, let us show that *the group $H = X \times G$ is not a PT-group*, thus answering the question of Pestov and Tkačenko in the negative.

Indeed, assume that H is a PT-group. Then μH is a topological group. Therefore, μH can be represented as a subgroup of Rajkov completion ρH containing H. Clearly, $\rho H = \rho X \times \rho G = \beta X \times G = \mu X \times \mu G$. Therefore, by Propositions 2.1 and 2.2, ρH is a minimal Dieudonné extension of H. Since $H \subset \mu H \subset \rho H$, it follows that $\mu H = \rho H$. However, H is C-embedded in μH. Hence, $H = X \times G$ is C-embedded in $\rho H = \beta X \times G$, which is not the case, as we have seen before. This contradiction completes the proof that H is not a PT-group.

From Theorems 1.2 and 1.8 it follows that the g-tightness of $\beta X \times G$ and of $H = X \times G$ is uncountable.

Several other observations about the construction in Example 2.6 are in order. First, for the role of the group X we may choose the Σ-product of ω_1 copies of the discrete group $D = \{0, 1\}$. Then βX is just the product D^{ω_1}, the weight of X and βX is ω_1, and the cardinality of G is ω_1. Notice that X in this case is countably compact and Fréchet-Urysohn.

Notice, that the space G is also hereditarily Hewitt-Nachbin complete, since every topological group of countable pseudocharacter can be mapped by a one-to-one continuous mapping onto a metrizable space (we take into account that the cardinality of G is Ulam non-measurable). Since X and G are strong PT-groups of Ulam non-measurable cardinality, it follows from Theorem 2.4 that $\upsilon X \times \upsilon G \neq \upsilon(X \times G)$ (and $\mu X \times \mu G \neq \mu(X \times G)$).

The group $\beta X \times G$ is not Moscow, since otherwise $X \times G$, as a dense subspace of $\beta X \times G$, would have been a Moscow space and, therefore, a strong PT-group. On the other hand, $\beta X \times G$ is, obviously, Rajkov complete and, hence, a strong PT-group. Thus, *a strong PT-group need not be a Moscow group, and a dense subgroup of a strong PT-group need not be a PT-group.*

We summarize the most important part of the information collected while we discussed Example 2.6 in the next two statements:

Theorem 2.7. *There exist a countably compact group X and a Rajkov complete group G of countable pseudocharacter with the following properties:*

1) *The product $X \times G$ is not a PT-group;*
2) *The product $\beta X \times G$ is not a Moscow group;*
3) $\mu X \times \mu G = \upsilon X \times \upsilon G \neq \upsilon(X \times G) = \mu(X \times G)$;
4) *The groups X, βX, and G are Moscow groups of countable g-tightness;*
5) *The g-tightness of $\beta X \times G$ (and of $X \times G$) is uncountable.*

Theorems 2.4 and 2.7 suggest that the notion of a strong PT-group has a role to play in the study of conditions under which the formula $\upsilon(G_1 \times G_2) = \upsilon G_1 \times \upsilon G_2$ holds. This natural question was given considerable attention, and a series of very interesting results in this direction were obtained in [12], [5], [6], [7] and [14]. In particular, it holds if $X \times Y$ is pseudocompact — this is a famous result of I. Glicksberg [12], reproved by Z. Frolík [10] by a different method.

Below we establish an amazingly general sufficient condition for the formula to hold in the case when X and Y are topological groups (see Theorem 2.9 and

Corollary 2.10). The key role again belongs to Moscow groups and strong PT-groups.

Proposition 2.8. *Let $G = \Pi\{G_\alpha : \alpha \in A\}$ be the topological product of topological groups G_α such that G is a strong PT-group and the cardinality of G is Ulam non-measurable. Then $\upsilon\Pi\{G_\alpha : \alpha \in A\} = \Pi\{\upsilon G_\alpha : \alpha \in A\}$.*

PROOF: We know that G_α is G_δ-dense in υG_α, for each $\alpha \in A$. Therefore, G is G_δ-dense in $\Pi\{\upsilon G_\alpha : \alpha \in A\}$. Each G_α is a strong PT-group, since it is C-embedded in G. Therefore, υG_α is a topological group, for each $\alpha \in A$, and $G^* = \Pi\{\upsilon G_\alpha : \alpha \in A\}$ is also a topological group.

Since G is a strong PT-group and G is G_δ-dense in G^*, it follows that G is C-embedded in $G^* = \Pi\{\upsilon G_\alpha : \alpha \in A\}$. Since $\Pi\{\upsilon G_\alpha : \alpha \in A\}$ is, obviously, Dieudonné complete and $|G|$ is Ulam non-measurable, it follows that $G^* = \Pi\{\upsilon G_\alpha : \alpha \in A\} = \upsilon G$. □

From Proposition 2.8 and Theorem 1.7 we obtain:

Theorem 2.9. *Let $G = \Pi\{G_\alpha : \alpha \in A\}$ be the topological product of topological groups G_α such that the space G is Moscow and $|G|$ is Ulam non-measurable. Then $\upsilon\Pi\{G_\alpha : \alpha \in A\} = \Pi\{\upsilon G_\alpha : \alpha \in A\}$.*

Corollary 2.10. *Let $\mathcal{F} = \{G_\alpha : \alpha \in A\}$ be a family of topological groups G_α such that the cardinality of the product group $G = \Pi\{G_\alpha : \alpha \in A\}$ is Ulam non-measurable. Then the formula*

$$\upsilon\Pi\{G_\alpha : \alpha \in A\} = \Pi\{\upsilon G_\alpha : \alpha \in A\}$$

holds if at least one of the following conditions is satisfied:

1) *Every group in $\mathcal{F}$ is totally bounded;*
2) *Every group in $\mathcal{F}$ is k-separable;*
3) *ω_1 is a precaliber of every space in $\mathcal{F}$;*
4) *The Souslin number of the product space $\Pi\{G_\alpha : \alpha \in A\}$ is countable;*
5) *The Souslin number of every group in $\mathcal{F}$ is countable, and $(MA + \neg CH)$ is satisfied;*
6) *Every group in $\mathcal{F}$ is κ-metrizable;*
7) *The tightness of the product space $\Pi\{G_\alpha : \alpha \in A\}$ is countable;*
8) *The g-tightness of the product group $\Pi\{G_\alpha : \alpha \in A\}$ is countable.*

PROOF: If the Souslin number of the product group G is countable, then G is Moscow. It remains to apply Theorem 2.9. This takes care of cases 1)–5). Similarly, in the cases 6), 7), and 8) the space G is also Moscow (see Theorem 1.8), and therefore Theorem 2.9 is applicable. □

Problem 2.11. Let G be a topological group of countable tightness. Is then the g-tightness of $G \times G$ countable? Is then $G \times G$ a Moscow group? A strong PT-group?

Notice, that we still do not have a ZFC-example of a topological group G of countable tightness such that the tightness of $G \times G$ is not countable.

It was observed in [3] that *every Lindelöf topological group is a strong PT-group, but not necessarily a Moscow space.* This can be deduced from another fact established in [3]: that every R-factorizable in the sense of Tkačenko group (see [20]) is a strong PT-group. Several further corollaries to the last statement and Proposition 2.8 are obtained in [3]. Also the list of conditions in Corollary 2.10 is expanded in [3].

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