On very weak solutions of a class of nonlinear elliptic systems

Menita Carozza, Antonia Passarelli di Napoli*

Abstract. In this paper we prove a regularity result for very weak solutions of equations of the type $-\operatorname{div} A(x,u,Du) = B(x,u,Du)$, where A, B grow in the gradient like t^{p-1} and $B(x,u,Du)$ is not in divergence form. Namely we prove that a very weak solution $u \in W^{1,r}$ of our equation belongs to $W^{1,p}$. We also prove global higher integrability for a very weak solution for the Dirichlet problem

$$\begin{cases} -\operatorname{div} A(x,u,Du) = B(x,u,Du) & \text{in } \Omega, \\ u - u_o \in W^{1,r}(\Omega,\mathbb{R}^m). \end{cases}$$

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1. Introduction

Let us consider equations of the type

$$-\operatorname{div} A(x,u,Du) = B(x,u,Du), \tag{1.1}$$

where $x \in \Omega$, a bounded open subset of $\mathbb{R}^n$, $n \geq 2$, $u : \Omega \longrightarrow \mathbb{R}^m$, $m \geq 1$ and $A : \Omega \times \mathbb{R}^m \times \mathbb{R}^{mn} \longrightarrow \mathbb{R}$ and $B : \Omega \times \mathbb{R}^m \times \mathbb{R}^{mn} \longrightarrow \mathbb{R}^n$ are Carathéodory functions such that

$$|\, A(x,u,z)\,| \leq c_1 + c_2|u|^{p-1} + c_3|z|^{p-1}, \tag{H1}$$

$$\langle\, A(x,u,z)\,, z\rangle \geq |z|^p - c_4|u|^p - c_5 \tag{H2}$$

and

$$|\, B(x,u,z)\,| \leq c_6 + c_7|u|^{p-1} + c_8|z|^{p-1}, \tag{H3}$$

where c_i, $i = 1,\dots,8$, and c are positive constants.

The previous assumptions allow us to give the following

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Definition 1.1. *A mapping* $u \in W^{1,r}_{loc}(\Omega, \mathbb{R}^m), \max\{1, p-1\} \le r < p$, *is called a very weak solution of the equation* (1.1) *if*

$$\int_\Omega [A(x,\, u,\, Du)D\Phi - B(x,\, u\, Du)\Phi]\, dx = 0$$

for all $\Phi \in W^{1,\frac{r}{r-p+1}}(\Omega, \mathbb{R}^m)$ *with compact support.*

The main result is the following

Theorem 1.2. *Let the assumptions* (H1)–(H3) *hold. Then there exists an exponent* $r_1 = r_1(m, n, p), \max\{1, p-1\} < r_1 < p$, *such that if* $u \in W^{1,r}_{loc}(\Omega, \mathbb{R}^m)$, $r_1 \le r < p$, *is a very weak solution of the equation* (1.1), *then* $u \in W^{1,p}_{loc}(\Omega, \mathbb{R}^m)$.

The theory of very weak solutions of equations of type (1.1) with the right hand-side in divergence form has been initiated by T. Iwaniec and C. Sbordone in [IS]. For that type of equations they proved that if r is sufficiently close to p, then a very weak solution really is a solution (see [I], [IS]). The main tool they used is the Hodge decomposition and later other authors used the same technique to approach similar problems (see [GLS], [M1]). In our case (the right hand-side of (1.1) is not in divergence form) the Hodge decomposition seems to be not useful. In proving Theorem 1.2 we follow the techniques of Lewis (see [Le], [M2]) using the theory about the Hardy-Littlewood maximal function and the A_p-weights. A fundamental tool in our proof is the choice of a suitable test function, involving level sets of maximal function defined by using a Lemma due to Acerbi and Fusco (see [AF] and Lemma 2.5 below). Another fundamental tool is a well known Hedberg estimate (see [H] and Lemma 2.6 below).

Remark 1.3. With the same techniques we can reobtain Theorem 1.2 for equations of the following type

$$-\operatorname{div}(w(x)\, A(x, u, Du)) \;=\; w(x)\, B(x, u, Du)$$

with $w(x)$ an A_p-weight (see [Mu] and Definition 2.1).

Remark 1.4. Note that the Euler-Lagrange system of the functional

$$I(u) = \int_\Omega [|Du|^p + |u|^p + a(x)]\, dx \tag{1.2}$$

is of type (1.1). Then Theorem 1.2 says also that a weak minimum of the functional (1.2) (see [IS], [M2]) really is a minimum. Instead for the general functional

$$I(u) = \int_\Omega f(x,\, u,\, Du)\, dx,$$

where f grows as $|Du|^p$, the Euler-Lagrange system has the right hand-side not in divergence form but growing with respect to the gradient as t^p. So that, unfortunately, Theorem 1.2 does not recover the previous general case.

Moreover, we consider the boundary value problem

$$(1.3)\qquad \begin{cases} -\operatorname{div} A(x,u,Du) = B(x,u,Du) & \text{in } \Omega \\ u - u_o \in W^{1,r}(\Omega,\mathbb{R}^m), \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary and A and B verify the assumptions (H1)–(H3). We will prove the global higher integrability of Du, with u solution of the problem (1.3). More precisely, we will prove the following:

Theorem 1.5. *Let* (H1)–(H3) *hold and assume* $u_o \in W^{1,p}(\Omega,\mathbb{R}^m)$. *Then there exists an exponent* $r_1 = r_1(m,n,p), \max\{1,p-1\} < r_1 < p$ *such that if* $u \in W^{1,r}(\Omega,\mathbb{R}^m)$, $r_1 \le r < p$, *is a very weak solution of the Dirichlet problem* (1.3), *then* $u \in W^{1,p}(\Omega,\mathbb{R}^m)$.

2. Preliminaries

In this section we introduce notations, definitions and preliminary results.

Let $B(x,r) = \{y \in \mathbb{R}^n : |y-x| < r\}$ and $|B(x,r)|$ denote its Lebesgue measure. For a measurable function f on $\mathbb{R}^n$ we set

$$f_{x,r} = \fint_{B(x,r)} |f(y)|\,dy = \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|\,dy.$$

Denote the Hardy-Littlewood maximal function of f by

$$Mf(x) = \sup_{r>0} \fint_{B(x,r)} |f(y)|\,dy$$

and set

$$M^k f(x) = M^{k-1}(Mf)(x) \quad \text{for } k \ge 2.$$

Definition 2.1. For $1 < p < \infty$, we say that a nonnegative measurable function $a \in L^1_{loc}(\mathbb{R}^n)$ is in the Muckenhoupt class A_p, or is an A_p-weight if and only if the quantity

$$A_p(a) = \sup_{x\in\mathbb{R}^n, r>0} \left(\fint_{B(x,r)} a\right)\left(\fint_{B(x,r)} a^{-\frac{1}{p-1}}\right)^{p-1}$$

is finite.

Now let us list some lemmas useful in the sequel.

Lemma 2.2. *Let $1 < p < \infty$. There exists a positive constant $c = c(n, p)$ such that for any $0 < 2\delta < p-1$, the function $(Mf)^{-\delta}$ is an A_p-weight and the quantity $A_p((Mf)^{-\delta})$ is less or equal to c for all $f \in L^1(\mathbb{R}^n)$, $f \neq 0$.*

For the proof see [Do], [Le] and [T].

We also recall the following well known theorem about A_p-weights (see [Mu])

Theorem 2.3. *For $1 < p < \infty$ and $a \in A_p$, there exists a positive constant $c = c(p, n, A_p(a))$ such that*

$$\int_{\mathbb{R}^n} a(x)(Mf(x))^p \, dx \leq c \int_{\mathbb{R}^n} a(x)|f(x)|^p \, dx$$

for all $f \in L^p(\mathbb{R}^n, a)$.

Moreover we will use the following lemmas.

Lemma 2.4. *Let $1 < p < \infty$, $x_0 \in \mathbb{R}^n$, $r > 0$ and $B = B(x_0, r)$. If $f \in W^{1,p}(B)$ then there exists $c = c(n, p)$ such that for any $x \in B$*

$$|f(x) - f_{x_0,r}| \leq c\, rM(|Df|\chi_B)(x),$$

where χ_B is the characteristic function of B.

Lemma 2.5. *Let $\lambda > 0$, $1 < q < \infty$, $x_0 \in \mathbb{R}^n$ and $r > 0$. Suppose $f \in W^{1,q}(\mathbb{R}^n)$,* $\operatorname{supp} f \subset B(x_0, r)$ *and*

$$F(\lambda) = \{x : M(|Df|)(x) \leq \lambda\} \cap B(x_0, 2r) \neq \phi.$$

Then $f_{/F(\lambda)}$ has an extension to $\mathbb{R}^n$, denoted by $v = v(\cdot, \lambda)$, such that

(i) *$v = f$ on $F(\lambda)$,*
(ii) $\operatorname{supp} v \subset B(x_0, 2r)$,
(iii) *$v \in W^{1,\infty}(\mathbb{R}^n)$ with $\|v\|_\infty \leq c\, \lambda\, r$ and $\|Dv\|_\infty \leq c\lambda$.*

Proof: See [AF] and [Le]. □

The following lemma is a result due to Hedberg (see [H]).

Lemma 2.6. *Let u be a function in $W_0^{1,p}(\Omega)$ and Ω a bounded open subset of $\mathbb{R}^n$. Set*

$$I(|Du|)(x) = \int_\Omega |Du|(y)|x - y|^{1-n} \, dy.$$

Then, the following estimate holds

$$u(x) \leq c\, I(|Du|)(x) \leq c\, M(|Du|)(x) \text{ a.e.}$$

where c is a positive constant depending on the dimension n and on the Lebesgue measure of Ω.

Proof: See [H] and [GT]. □

Finally, we need the theorem (see [G] and [Gi])

Theorem 2.7. *Let $R > 0$, $q > 1$ and $g \in L^q(B(x_0, R))$ be such that*

$$\fint_{B(x,\frac{r}{8})} |g|^q \, dx \le c\Big(\fint_{B(x,r)} |g| \, dx\Big)^q + \vartheta \fint_{B(x,r)} |g|^q \, dx + \tilde{c}$$

for $0 < \vartheta < 1$ and $x \in B(x_0, R/2)$, $0 < r \le R/8$.

Then there exists $c' = c'(n, \vartheta, c, q)$ and $\eta = \eta(n, \vartheta, c, q) > 0$ such that if $\tau = q(1 + \eta)$ then

$$\Big(\fint_{B(x,R/4)} |g|^\tau \, dx\Big)^{\frac{1}{\tau}} \le c'\Big(\fint_{B(x,R/2)} |g|^q \, dx\Big)^{1/q} + \tilde{c}.$$

3. Main results

Proof of Theorem 1.2. Let $B = B(x_0, R) \subset \Omega$ for some $R \le 1$. For fixed $y_0 \in B(x_0, R/2)$ and $0 < \rho < R/8$, let $B\rho = B(y_0, \rho)$ and $\varphi \in C_0^\infty(B_{2\rho})$ be such that $\varphi = 1$ on $B\rho$, $0 \le \varphi \le 1$ on $B_{2\rho}$ and $|D\varphi| \le c\,\rho^{-1}$.

With $u_{4\rho} = \fint_{B_{4\rho}} u(x)\, dx$, we set $\tilde{u} = (u - u_{4\rho})\varphi$, $E(\lambda) = \{x \in \mathbb{R}^n : M(|D\tilde{u}|) \le \lambda\}$ and $F_\lambda = E_\lambda \cap B_{4\rho}$.

Since supp $\tilde{u} \subset B_{2\rho}$, we observe that for $x \in \mathbb{R}^n - B_{3\rho}$

$$M(|D\tilde{u}|)(x) \le c\,\rho^{-n} \int_{B_{2\rho}} |D\tilde{u}|(y)\, dy, \tag{3.1}$$

where c is a constant depending only on the dimension n, and setting

$$\lambda_0 = c\,\rho^{-n} \int_{B_{2\rho}} |D\tilde{u}|(y)\, dy,$$

$F(\lambda)$ is not empty for $\lambda > \lambda_0$ and thanks to Lemma 2.5 we can extend the function $\tilde{u}_{|F(\lambda)}$ to whole $\mathbb{R}^n$.

Let v be the extension of $\tilde{u}_{|F(\lambda)}$. v satisfies the conditions (i)–(iii) (see Lemma 2.5) so that we can consider v as a particular test function in Definition 1.1. By (H1) and (H3) we get

$$\begin{aligned}
&\int_{F(\lambda)} [A(x,\, u,\, Du)\, D\tilde{u} - B(x,\, u,\, Du)\, \tilde{u}]\, dx \\
&= \int_{B_{4\rho} - F(\lambda)} [B(x,\, u,\, Du)\, v - A(x,\, u,\, Du)\, Dv\,]\, dx \\
&\le c\,\lambda \int_{B_{4\rho} - F(\lambda)} [|Du|^{p-1} + |u|^{p-1} + 1] + \rho[|Du|^{p-1} + |u|^{p-1} + 1]\, dx.
\end{aligned}$$

Multiplying both sides of the previous inequality by $\lambda^{-(1+\delta)}$, where $\delta = p - r$ will be chosen at the end of the proof, and integrating from λ_0 to $+\infty$, we have

$$\int_{\lambda_0}^{+\infty} \lambda^{-(1+\delta)}\, d\lambda \int_{B_{4\rho}} [A(x,\, u,\, Du)\, D\tilde{u} \,-\, B(x,\, u,\, Du)\, \tilde{u}]\, \chi_{\{M(|D\tilde{u}|)\le\lambda\}}\, dx \tag{3.2}$$
$$\le c \int_{\lambda_0}^{+\infty} \lambda^{-\delta}\, d\lambda \int_{B_{4\rho}-F(\lambda)} [(|Du|^{p-1}+|u|^{p-1}+1)+\rho(|Du|^{p-1}+|u|^{p-1}+1)]\, dx.$$

Interchanging the order of integration, the left hand side of (3.2) becomes

$$\int_{B_{4\rho}-E(\lambda_0)} [A(x,\, u,\, Du)\, D\tilde{u} \,-\, B(x,\, u,\, Du)\, \tilde{u}]\, dx \int_{M(|D\tilde{u}|)}^{+\infty} \lambda^{-(1+\delta)}\, d\lambda$$
$$+ \int_{\lambda_0}^{+\infty} \lambda^{-(1+\delta)}\, d\lambda \int_{E(\lambda_0)} [A(x,\, u,\, Du)\, D\tilde{u} \,-\, B(x,\, u,\, Du)\, \tilde{u}]\, dx$$
$$= \frac{1}{\delta} \int_{B_{4\rho}-E(\lambda_0)} [A(x,\, u,\, Du)\, D\tilde{u} \,-\, B(x,\, u,\, Du)\, \tilde{u}]\, M(|D\tilde{u}|)^{-\delta}\, dx \tag{3.3}$$
$$+ \frac{\lambda_0^{-\delta}}{\delta} \int_{E(\lambda_0)} [A(x,\, u,\, Du)\, D\tilde{u} \,-\, B(x,\, u,\, Du)\, \tilde{u}]\, dx$$
$$\equiv \frac{1}{\delta} J_1 + \frac{\lambda_0^{-\delta}}{\delta} J_2.$$

Let us recall that $\operatorname{supp} \tilde{u} \subset B_{2\rho}$, $\tilde{u} = u$ on B_ρ and $B_{4\rho} - E(\lambda_0) = B_{4\rho} - F(\lambda_0)$, so we have

$$J_1 = \int_{B_{4\rho}} [A(x,\, u,\, Du)]\, D\tilde{u} \,-\, B(x,\, u,\, Du)\, \tilde{u}]\, M(|D\tilde{u}|)^{-\delta}\, dx$$
$$- \int_{F(\lambda_0)} [A(x,\, u,\, Du)]\, D\tilde{u} \,-\, B(x,\, u,\, Du)\, \tilde{u}]\, M(|D\tilde{u}|)^{-\delta}\, dx$$
$$= \int_{B_{2\rho}-B_\rho} [A(x,\, u,\, Du)\, D\tilde{u} \,-\, B(x,\, u,\, Du)\, \tilde{u}]\, M(|D\tilde{u}|)^{-\delta}\, dx \tag{3.4}$$
$$- \int_{F(\lambda_0)} [A(x,\, u,\, Du)\, D\tilde{u} \,-\, B(x,\, u,\, Du)\, \tilde{u}]\, M(|D\tilde{u}|)^{-\delta}\, dx$$
$$+ \int_{B_\rho} [A(x,\, u,\, Du)\, Du \,-\, B(x,\, u,\, Du)\, u]\, M(|D\tilde{u}|)^{-\delta}\, dx.$$

By (3.2), (3.3) and (3.4) we obtain

$$\frac{1}{\delta} \int_{B_\rho} [A(x,\, u,\, Du)\, Du \,-\, B(x,\, u,\, Du)\, u]\, M(|D\tilde{u}|)^{-\delta}\, dx$$
$$\le \frac{1}{\delta} \int_{F(\lambda_0)} [A(x,\, u,\, Du)\, D\tilde{u} \,-\, B(x,\, u,\, Du)\, \tilde{u}]\, M(|D\tilde{u}|)^{-\delta}\, dx$$

$$+\frac{1}{\delta}\int_{B_{2\rho}-B_\rho}[B(x,\,u,\,Du)\,\tilde{u}-A(x,\,u,\,Du)\,D\tilde{u}\,]\,M(|D\tilde{u}|)^{-\delta}\,dx$$

$$+\frac{\lambda_0^{-\delta}}{\delta}\int_{E(\lambda_0)\cap B_{2\rho}}[B(x,\,u,\,Du)\,\tilde{u}-A(x,\,u,\,Du)\,D\tilde{u}\,]\,dx$$

$$+c\int_{\lambda_0}^{+\infty}\lambda^{-\delta}d\lambda\int_{B_{4\rho}-F(\lambda)}[|Du|^{p-1}+|u|^{p-1}+1]\,dx.$$

Moreover, since $\lambda_0^{-\delta}\le M(|D\tilde{u}|)^{-\delta}$ on $E(\lambda_0)$, using (H1),(H2),(H3) and multiplying by δ we obtain

$$\int_{B\rho}(|Du|^p)M(|D\tilde{u}|)^{-\delta}\,dx$$
$$\le c\int_{E(\lambda_0)\cap B_{2\rho}}|(D\tilde{u}+\tilde{u})|(|Du|^{p-1}+|u|^{p-1}+1)M(|D\tilde{u}|)^{-\delta}\,dx$$
$$+c\int_{B_{2\rho}-B_\rho}(|D\tilde{u}||Du|^{p-1}+|D\tilde{u}||u|^{p-1}+|D\tilde{u}|)M(|D\tilde{u}|)^{-\delta}\,dx$$
$$+c\int_{B_{2\rho}}(|\tilde{u}||Du|^{p-1}+|\tilde{u}||u|^{p-1}+|\tilde{u}|+c)M(|D\tilde{u}|)^{-\delta}\,dx$$
$$+c\delta\int_{\lambda_0}^{+\infty}\lambda^{-\delta}d\lambda\int_{B_{4\rho}}(|Du|^{p-1}+|u|^{p-1}+1)\chi_{\{M(|D\tilde{u}|)>\lambda\}}\,dx.$$

We write the previous relation as

$$I_0\le c[I_1+I_2+I_3]+c\delta I_4. \tag{3.5}$$

To simplify the presentation we will estimate the integrals I_i, $i=1,2,3,4$ at the end of this section.

<u>Conclusion</u>.

By the estimates of the integrals I_i below, we get

$$\begin{aligned} I_0 &\le c\left(\eta^{1-\delta}+\delta^{1-\delta}+\frac{\delta}{1-\delta}\right)\int_{B_{4\rho}}|Du|^{p-\delta}\,dx\\ &\quad+c(\eta^{1-p}+\eta^{\frac{1}{1-p}}+\delta^{-\delta})\rho^n\left(\fint_{B_{4\rho}}|Du|^t\right)^{\frac{p-\delta}{t}}\\ &\quad+c\delta^{-\delta}\int_{B_{2\rho}\setminus B_{\frac{\rho}{2}}}|Du|^{p-\delta}\,dx+c\rho^n. \end{aligned} \tag{3.6}$$

Observe that by Lemma 2.4

$$|u(x)-u_{4\rho}|\le c\rho[M(|Du|\chi_{B_{4\rho}})]\quad\text{for any}\quad x\in B_{4\rho}$$

and then

$$|D\tilde{u}| \le |Du| + c[M(|Du|\chi_{B_{4\rho}})]. \tag{3.7}$$

Since $\tilde{u} = u$ on B_ρ, we see that for $x \in B_{\frac{\rho}{2}}$

$$\begin{aligned} M(|D\tilde{u}|) &\le M(|Du|\chi_{B_\rho}) + c \fint_{B_{4\rho}} |D\tilde{u}|\, dx \\ &\le M(|Du|\chi_{B_\rho}) + c \fint_{B_{4\rho}} [M(|Du|\chi_{B_{4\rho}})]\, dx. \end{aligned}$$

On the other hand, setting

$$H = \ \{x \in B_{\frac{\rho}{2}} :\ M(|Du|\chi_{B_\rho})(x) \ge c \fint_{B_{4\rho}} M(|Du|\chi_{B_{4\rho}})(x)\, dx\}$$

we have

$$M(|D\tilde{u}|)(x) \le cM(|Du|\chi_{B_\rho})(x) \ \text{ on } \ H.$$

Then

$$\begin{aligned} &\int_{B_\rho} |Du|^p M(|D\tilde{u}|)^{-\delta} \ge c \int_{B_\rho} M(|Du|\chi_{B_\rho})^p M(|D\tilde{u}|)^{-\delta} \\ &\ge c \int_H M(|Du|\chi_{B_\rho})^p M(|D\tilde{u}|)^{-\delta} \ge c \int_H M(|Du|\chi_{B_\rho})^p M(|Du|\chi_{B_\rho})^{-\delta}\, dx \\ &= c \int_{B_{\frac{\rho}{2}}} M(|Du|\chi_{B_\rho})^{p-\delta}\, dx - c \int_{B_{\frac{\rho}{2}} \setminus H} M(|Du|\chi_{B_\rho})^{p-\delta}\, dx \\ &\ge c \int_{B_{\frac{\rho}{2}}} |Du|^{p-\delta} - c\rho^n \left(\fint_{B_{4\rho}} M(|Du|\chi_{B_{4\rho}})\, dx \right)^{p-\delta} \\ &\ge c \int_{B_{\frac{\rho}{2}}} |Du|^{p-\delta} - c\rho^n \left(\fint_{B_{4\rho}} M(|Du|\chi_{B_{4\rho}})^t\, dx \right)^{\frac{p-\delta}{t}} \\ &\ge c \int_{B_{\frac{\rho}{2}}} |Du|^{p-\delta} - c\rho^n \left(\fint_{B_{4\rho}} |Du|^t\, dx \right)^{\frac{p-\delta}{t}}, \end{aligned}$$

where we applied Lemma 2.2 and Muckenhoupt's Theorem in the first and last inequality, in previous estimate. Since we will apply Sobolev-Poincaré inequality in the estimates of I_i, we have to choose $(p-\delta)_* \le t \le p-\delta$, where as usual $(p-\delta)_* = \frac{n(p-\delta)}{n+p-\delta}$. Then we have

$$\begin{aligned} I_0 &= \int_{B_\rho} |Du|^p M(|D\tilde{u}|)^{-\delta} \\ &\ge c \int_{B_{\frac{\rho}{2}}} |Du|^{p-\delta} - c\rho^n \left(\fint_{B_{4\rho}} |Du|^t\, dx \right)^{\frac{p-\delta}{t}}. \end{aligned} \tag{3.8}$$

From inequalities (3.6) and (3.8) it follows that

$$
\begin{aligned}
&\int_{B_{\frac{\rho}{2}}} |Du|^{p-\delta}\,dx \\
&\le c\Big(\eta^{1-\delta}+\delta^{1-\delta}+\frac{\delta}{1-\delta}\Big)\int_{B_{4\rho}}|Du|^{p-\delta}\,dx \\
&+ c(\eta^{1-p}+\delta^{-\delta}+\eta^{\frac{1}{1-p}})\rho^n\Big(\fint_{B_{4\rho}}|Du|^t\Big)^{\frac{p-\delta}{t}} \\
&+ c\delta^{-\delta}\int_{B_{2\rho}\setminus B_{\frac{\rho}{2}}}|Du|^{p-\delta}\,dx + c\rho^n.
\end{aligned}
$$

Now, applying the “hole filling”, we add the quantity

$$
c\,\delta^{-\delta}\int_{B_{\frac{\rho}{2}}}|Du|^{p-\delta}\,dx
$$

to both sides of the previous inequality and we get

$$
\begin{aligned}
&\fint_{B_{\frac{\rho}{2}}} |Du|^{p-\delta}\,dx \\
&\le \frac{c}{c\delta^{-\delta}+1}\Big(\eta^{1-\delta}+\delta^{-\delta}+\delta^{1-\delta}+\frac{\delta}{1-\delta}\Big)\fint_{B_{4\rho}}|Du|^{p-\delta}\,dx \\
&+ \hat{c}\Big(\fint_{B_{4\rho}}|Du|^t\Big)^{\frac{p-\delta}{t}} + \tilde{c}.
\end{aligned}
$$

Notice that there exist $0<\delta_1<1$ and $0<\eta_1<1$ such that if $0<\delta<\delta_1$ and $0<\eta<\eta_1$,

$$
\frac{c}{c\delta^{-\delta}+1}\Big(\eta^{1-\delta}+\delta^{-\delta}+\delta^{1-\delta}+\frac{\delta}{1-\delta}\Big)\le\vartheta<1.
$$

From the estimates above we have for $0<\delta<\delta_1$ and $0<\eta<\eta_1$

$$
\begin{aligned}
&\fint_{B_{\rho/2}} |Du|^{p-\delta}\,dx \\
&\le \vartheta\fint_{B_{4\rho}}|Du|^{p-\delta}\,dx + \hat{c}\Big(\fint_{B_{4\rho}}|Du|^t\,dx\Big)^{\frac{p-\delta}{t}} + \tilde{c},
\end{aligned}
$$

where $\hat{c}$ depends on m,n,p but not on δ.

The result follows from Theorem 2.6 with an argument similar to the one of [GLS].

Now let us estimate the integrals $I_i,\ i=1,2,3,4$.

Estimate of I_1.

$$I_1=\int_{E(\lambda_0)\cap B_{2\rho}}(|D\tilde u|+|\tilde u|)(|Du|^{p-1}+|u|^{p-1}+1)M(|D\tilde u|)^{-\delta}\,dx$$
$$\le c\int_{E(\lambda_0)\cap B_{2\rho}}(|Du|^{p-1}+|u|^{p-1}+1)M(|D\tilde u|)^{1-\delta}\,dx$$

by Lemma 2.6.

Let us suppose $0<\eta\le\frac{1}{2}$ and $|Du|\ge\eta^{-1}\lambda_0$, then at $x\in E(\lambda_0)$ we have

$$M(|D\tilde u|)\le\lambda_0\le|Du|\eta \tag{3.9}$$

and, therefore,

$$|Du|^{p-1}M(|D\tilde u|)^{1-\delta}\le\eta^{1-\delta}|Du|^{p-\delta}. \tag{3.10}$$

On the other hand, if $x\in E(\lambda_0)$ and $|Du|<\eta^{-1}\lambda_0$ we get

$$|Du|^{p-1}M(|D\tilde u|)^{1-\delta}\le\eta^{1-p}\lambda_0^{p-\delta}. \tag{3.11}$$

Then by (3.10), (3.11) in $E(\lambda_0)\cap B_{2\rho}$ we have

$$|Du|^{p-1}M(|D\tilde u|)^{1-\delta}\le c(\eta^{1-p}\lambda_0^{p-\delta}+\eta^{1-\delta}|Du|^{p-\delta}).$$

By the definition of λ_0 and formula (3.7), we note that

$$\begin{aligned}\eta^{1-p}\lambda_0^{p-\delta}&\le c\,\eta^{1-p}\left(\fint_{B_{4\rho}}M(|Du|\chi_{B_{4\rho}})\,dx\right)^{p-\delta}\\&\le c\eta^{1-p}\left(\fint_{B_{4\rho}}M(|Du|\chi_{B_{4\rho}})^t\,dx\right)^{\frac{p-\delta}{t}},\end{aligned} \tag{3.12}$$

where $(p-\delta)_*=\frac{n(p-\delta)}{n+p-\delta}\le t<p-\delta$. Finally, by the estimates above and the Hardy-Littlewood theorem we get

$$I_1\le c\,\eta^{1-\delta}\int_{B_{4\rho}}|Du|^{p-\delta}\,dx+c\,\eta^{1-p}\rho^n\left(\fint_{B_{4\rho}}|Du|^t\,dx\right)^{\frac{p-\delta}{t}}$$
$$+\int_{E(\lambda_0)\cap B_{2\rho}}(|u|^{p-1}+1)M(|D\tilde u|)^{1-\delta}\,dx.$$

On the other hand, for $0 < \eta \le \frac{1}{2}$ and $|u| \ge \eta^{-1}\lambda_0$, we have for $x \in E(\lambda_0)$

$$|u|^{p-1} M(|D\tilde{u}|)^{1-\delta} \le |u|^{p-\delta} \eta^{1-\delta} \lambda_0^{\delta-1} M(|D\tilde{u}|)^{1-\delta} \le \eta^{1-\delta} |u|^{p-\delta}.$$

If $|u| < \eta^{-1}\lambda_0$, we have

$$|u|^{p-1} M(|D\tilde{u}|)^{1-\delta} \le c\eta^{1-p} \lambda_0^{p-1} \lambda_0^{1-\delta} = c\eta^{1-p} \lambda_0^{p-\delta}.$$

Therefore, by estimate (3.12) above,

$$\begin{aligned}\int_{E(\lambda_0)\cap B_{2\rho}} &|u|^{p-1} M(|D\tilde{u}|)^{1-\delta} \\ &\le c\eta^{1-p}\rho^n \biggl(\fint_{B_{4\rho}} |Du|^t \, dx \biggr)^{\frac{p-\delta}{t}} + c\eta^{1-\delta} \int_{E(\lambda_0)\cap B_{2\rho}} |u|^{p-\delta}\end{aligned}$$

with $t < p - \delta$. Moreover using Young inequality we have that

$$\begin{aligned}\int_{E(\lambda_0)\cap B_{2\rho}} M(|D\tilde{u}|)^{1-\delta} \, dx &\le \int_{B_{4\rho}} M(|D\tilde{u}|)^{1-\delta} \, dx \\ &\le c\eta^{1-\delta} \int_{B_{4\rho}} M(|D\tilde{u}|)^{p-\delta} \, dx + c\eta^{\frac{-(1-\delta)^2}{p-1}} \rho^n \\ &\le c\eta^{1-\delta} \int_{B_{4\rho}} [M^2(|Du\chi_{B_{4\rho}}|)]^{p-\delta} \, dx + c\eta^{\frac{1}{1-p}} \rho^n \\ &\le c\eta^{1-\delta} \int_{B_{4\rho}} |Du|^{p-\delta} \, dx + c\eta^{\frac{1}{1-p}} \rho^n.\end{aligned}$$

Therefore

$$I_1 \le c\eta^{1-p}\rho^n \biggl(\fint_{B_{4\rho}} |Du|^t \, dx \biggr)^{\frac{p-\delta}{t}} + c\eta^{1-\delta} \int_{B_{4\rho}} |Du|^{p-\delta} \, dx + c\eta^{\frac{1}{1-p}} \rho^n. \tag{3.13}$$

Estimate of I_2.

We have now to estimate the integral

$$\begin{aligned} I_2 \le &\int_{B_{2\rho}\setminus B_\rho} |D\tilde{u}||Du|^{p-1} M(|D\tilde{u}|)^{-\delta} \, dx \\ &+ \int_{B_{2\rho}\setminus B_\rho} |D\tilde{u}||u|^{p-1} M(|D\tilde{u}|)^{-\delta} \, dx \\ &+ \int_{B_{2\rho}\setminus B_\rho} |D\tilde{u}| M(|D\tilde{u}|)^{-\delta} \, dx = c(J + JJ + JJJ).\end{aligned} \tag{3.14}$$

Let D_1 be the set of all $x \in B_{2\rho} \setminus B_\rho$ such that

$$M(|D\tilde{u}|)(x) \le \delta M(|Du|\chi_{B_{4\rho}})(x)$$

and set $D_2 = (B_{2\rho} - B_\rho) - D_1$. Then

$$\begin{aligned} J \le & \int_{D_1} |D\tilde{u}||Du|^{p-1} M(|D\tilde{u}|)^{-\delta}\, dx + \int_{D_2} |\varphi||Du|^p M(|D\tilde{u}|)^{-\delta}\, dx \\ & + \frac{c}{\rho} \int_{D_2} |u - u_{4\rho}||Du|^{p-1} M(|D\tilde{u}|)^{-\delta}\, dx. \end{aligned}$$

Next, from the definition of D_1 and the Hardy-Littlewood maximal theorem, we get

$$\begin{aligned} & \int_{D_1} |D\tilde{u}|\; |Du|^{p-1} M(|D\tilde{u}|)^{-\delta}\, dx \\ \le & \int_{D_1} M(|D\tilde{u}|)^{1-\delta} |Du|^{p-1}\, dx \le c\delta^{1-\delta} \int_{B_{4\rho}} |Du|^{p-\delta}\, dx. \end{aligned}$$

On the other hand, since $M(|Du|\chi_{B_{4\rho}})(x) \ge (|Du|\chi_{B_{4\rho}})(x)$, we have

$$\begin{aligned} & \int_{D_2} |\varphi|\; |Du|^p M(|D\tilde{u}|)^{-\delta}\, dx \\ \le \; & \delta^{-\delta} \int_{D_2} |Du|^{p-\delta}\, dx \le \delta^{-\delta} \int_{B_{2\rho} - B_\rho} |Du|^{p-\delta}\, dx. \end{aligned}$$

Finally, by Young's inequality, we obtain

$$\begin{aligned} & \int_{D_2} \frac{|u - u_{4\rho}|}{\rho} |Du|^{p-1} M(|D\tilde{u}|)^{-\delta}\, dx \le \delta^{-\delta} \int_{D_2} \frac{|u - u_{4\rho}|}{\rho} |Du|^{p-1-\delta}\, dx \\ \le \; & \delta^{-\delta} \int_{D_2} |Du|^{p-\delta}\, dx + c \int_{B_{4\rho}} \left(\frac{|u - u_{4\rho}|}{\rho} \right)^{p-\delta} dx \\ \le \; & \delta^{-\delta} \int_{B_{2\rho} - B_\rho} |Du|^{p-\delta}\, dx + c\, \rho^n \left(\fint_{B_{4\rho}} |Du|^t\, dx \right)^{\frac{p-\delta}{t}}, \end{aligned}$$

where $(p-\delta)_* = \frac{n(p-\delta)}{n+p-\delta} \le t < p - \delta$.

Then, by the previous estimates we can conclude that

$$\begin{aligned} J \le \; & c\, \delta^{1-\delta} \int_{B_{4\rho}} |Du|^{p-\delta}\, dx \\ & + c\, \delta^{-\delta} \int_{B_{2\rho} - B_\rho} |Du|^{p-\delta}\, dx + c\, \rho^n \left(\fint_{B_{4\rho}} |Du|^t\, dx \right)^{\frac{p-\delta}{t}}. \end{aligned} \tag{3.15}$$

To estimate JJ we remark that by Young inequality and (3.7)

$$
\begin{aligned}
JJ &\le \int_{B_{2\rho}\setminus B_\rho} |u|^{p-1} M(|D\tilde{u}|)^{1-\delta}\,dx \\
&\le c\eta^{1-\delta} \int_{B_{2\rho}\setminus B_\rho} M(|D\tilde{u}|)^{p-\delta}\,dx + c\eta^{\frac{-(1-\delta)^2}{p-1}} \left(\int_{B_{2\rho}\setminus B_\rho} |u|^{p-\delta}\,dx \right) \\
&\le c\eta^{1-\delta} \int_{B_{2\rho}\setminus B_\rho} [M^2(|Du\chi_{B_{4\rho}}|)]^{p-\delta}\,dx + c\eta^{\frac{1}{1-p}} \left(\int_{B_{2\rho}\setminus B_\rho} |u|^{p-\delta}\,dx \right) \\
&\le c\eta^{1-\delta} \int_{B_{4\rho}} |Du|^{p-\delta}\,dx + c\eta^{\frac{1}{1-p}} \left(\int_{B_{2\rho}\setminus B_\rho} |u|^{p-\delta}\,dx \right) \\
&\le c\eta^{1-\delta} \int_{B_{4\rho}} |Du|^{p-\delta}\,dx + c\eta^{\frac{1}{1-p}} \rho^n \left(⨍_{B_{4\rho}} |Du|^t\,dx \right)^{\frac{p-\delta}{t}},
\end{aligned}
\tag{3.16}
$$

where $0 < \eta < \frac{1}{2}$. Arguing as in the previous estimate we have

$$
\begin{aligned}
JJJ &\le \int_{B_{2\rho}\setminus B_\rho} M(|D\tilde{u}|)^{1-\delta}\,dx \\
&\le c\eta^{1-\delta} \int_{B_{2\rho}\setminus B_\rho} M(|D\tilde{u}|)^{p-\delta}\,dx + c\eta^{\frac{-(1-\delta)^2}{p-1}} \rho^n \\
&\le c\eta^{1-\delta} \int_{B_{2\rho}\setminus B_\rho} [M^2(|Du\chi_{B_{4\rho}}|)]^{p-\delta}\,dx + c\eta^{\frac{1}{1-p}} \rho^n \\
&\le c\eta^{1-\delta} \int_{B_{4\rho}} |Du|^{p-\delta}\,dx + c\eta^{\frac{1}{1-p}} \rho^n.
\end{aligned}
\tag{3.17}
$$

Then from (3.15), (3.16), (3.17) we get

$$
\begin{aligned}
I_2 &\le c(\delta^{1-\delta} + \eta^{1-\delta}) \int_{B_{4\rho}} |Du|^{p-\delta}\,dx + c\eta^{\frac{1}{1-p}} \rho^n \left(⨍_{B_{4\rho}} |Du|^t\,dx \right)^{\frac{p-\delta}{t}} \\
&\quad + c\delta^{-\delta} \int_{B_{2\rho}\setminus B_\rho} |Du|^{p-\delta}\,dx + c\eta^{\frac{1}{1-p}} \rho^n.
\end{aligned}
\tag{3.18}
$$

Estimate of I_3.

Using Lemma 2.6 and Young's inequality we have that

$$
\begin{aligned}
I_3 &\leq \int_{B_{2\rho}} (|\tilde{u}||Du|^{p-1} + |\tilde{u}||u|^{p-1} + |\tilde{u}|)M(|D\tilde{u}|)^{-\delta}\,dx \\
&\leq \int_{B_{2\rho}} (|\tilde{u}|^{1-\delta}|Du|^{p-1} + |\tilde{u}|^{p-\delta} + |\tilde{u}|^{1-\delta})\,dx \\
&\leq c\eta^{1-\delta}\int_{B_{2\rho}} (|D\tilde{u}|)^{p-\delta}\,dx + c(\eta^{\frac{-(1-\delta)^2}{p-1}} + 1)\left(\int_{B_{2\rho}} |\tilde{u}|^{p-\delta}\,dx\right) + c\rho^n \qquad (3.19)\\
&\leq c\eta^{1-\delta}\int_{B_{4\rho}} |Du|^{p-\delta}\,dx + c(\eta^{\frac{1}{1-p}} + 1)\left(\int_{B_{2\rho}} |u|^{p-\delta}\,dx\right) + c\rho^n \\
&\leq c\eta^{1-\delta}\int_{B_{4\rho}} |Du|^{p-\delta}\,dx + c\eta^{\frac{1}{1-p}}\rho^n\left(\fint_{B_{4\rho}} |Du|^t\,dx\right)^{\frac{p-\delta}{t}} + c\rho^n,
\end{aligned}
$$

where $0 < \eta < \frac{1}{2}$.

Estimate of I_4.

By using Lemma (2.6) and the Hardy-Littlewood maximal theorem, we get

$$
\begin{aligned}
I_4 &= \int_{B_{4\rho}} |Du|^{p-1} + |u|^{p-1}\left(\int_{\lambda_0}^{M(|D\tilde{u}|)} \lambda^{-\delta}d\lambda\right)\,dx \\
&\leq \frac{1}{1-\delta}\int_{B_{4\rho}} |Du|^{p-1}M(|D\tilde{u}|)^{1-\delta}\,dx + \frac{1}{1-\delta}\int_{B_{4\rho}} |u|^{p-1}M(|D\tilde{u}|)^{1-\delta}dx \qquad (3.20)\\
&\leq \frac{c}{1-\delta}\int_{B_{4\rho}} |Du|^{p-\delta}\,dx + \frac{c}{1-\delta}\int_{B_{4\rho}} |u|^{p-\delta}\,dx \\
&\leq \frac{c}{1-\delta}\int_{B_{4\rho}} |Du|^{p-\delta}\,dx.
\end{aligned}
$$

Proof of Theorem 1.5. First, let us remark that we have only to prove the regularity near the boundary $\partial\Omega$, since the local higher integrability result has been proved in Theorem 1.2. For $z \in \mathbb{R}^n$, let us introduce the following notations:

$$
\begin{aligned}
Q_R(z) &= \{x \in \mathbb{R}^n : |x_i - z_i| < R,\ i = 1, \dots, n\}, \\
Q_R^+(z) &= \{x \in Q_R(z) : x_n > 0\}, \\
Q_R^-(z) &= \{x \in Q_R(z) : x_n < 0\}, \\
\Gamma_R(z) &= \{x \in Q_R(z) : x_n = 0\}.
\end{aligned}
$$

The compactness of $\bar{\Omega}$ implies that it is possible to recover $\partial\Omega$ with a finite number of neighborhoods V of its points. For every such neighborhood V, there exists a Lipschitz continuous function G, with Lipschitz inverse, such that

$$G(V) = Q_1(0),\ \ G(V\cap\Omega) = Q_1^+(0),\ \ G(V\cap\mathbb{R}^n\setminus\bar{\Omega}) = Q_1^-(0),\ \ G(V\cap\partial\Omega) = \Gamma_1(0).$$

Setting $\bar{u}(y) = u(G^{-1}(y))$, it is standard to prove that $\bar{u}$ solves the equation

$$\int_{Q^+} \mathcal{A}(x,\bar{u},D\bar{u})D\Phi\,dx = \int_{Q^+} \mathcal{B}(x,\bar{u},D\bar{u})\Phi\,dx \qquad \forall\,\Phi \in W^{1,\frac{r}{r-p+1}}(Q^+),$$

where $\mathcal{A}$, $\mathcal{B}$ are Carathéodory functions which verify the assumptions (H1)–(H3). Let us consider $x_0 \in \partial\Omega$ and a cube $Q = Q(x_0,R)$ for some $R \le 1$. For fixed $y_0 \in Q(x_0,R/2)$ and $0 < \rho < R/8$, let $Q\rho = B(y_0,\rho)$ and $\varphi \in C_0^\infty(Q_{2\rho})$ be such that $\varphi = 1$ on $Q\rho$, $0 \le \varphi \le 1$ on $Q_{2\rho}$ and $|D\varphi| \le c\,\rho^{-1}$.

With $(\bar{u}-\bar{u}_o)_{4\rho} = \fint_{Q_{4\rho}} \bar{u}(x) - \bar{u}_o(x)\,dx$, we set $\tilde{w} = ((\bar{u}-\bar{u}_o) - (\bar{u}-\bar{u}_o)_{4\rho})\varphi$, $E(\lambda) = \{x \in \mathbb{R}^n : M(|D\tilde{w}|) \le \lambda\}$ and $F_\lambda = E_\lambda \cap Q_{4\rho}$.

Since $\operatorname{supp}\tilde{w} \subset Q_{2\rho}$, for $x \in \mathbb{R}^n - Q_{3\rho}$ we observe that

$$M(|D\tilde{w}|)(x) \le c\,\rho^{-n}\int_{Q_{2\rho}} |D\tilde{w}|(y)\,dy = \lambda_0.$$

$F(\lambda)$ is not empty for $\lambda > \lambda_0$ and thanks to Lemma 2.5 we can extend the function $\tilde{w}_{|F(\lambda)}$ to whole $\mathbb{R}^n$.

Let Φ be the extension of $\tilde{w}_{|F(\lambda)}$. Φ satisfies the conditions (i)–(iii) (see Lemma 2.5) so that we can consider Φ as a particular test function. After the choice of that test function the proof can be achieved arguing as in Theorem 1.2.

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Universitá degli Studi di Salerno, Via S. Allende, 84081 Baronissi (SA) Italy

Universitá degli Studi di Napoli "Federico II", Complesso M. Sant' Angelo, 80126 Napoli, Italy

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