Equivalence of the properties (β) and (NUC) in Orlicz spaces

Sompong Dhompongsa

Abstract. We obtain the equivalence of the properties (β) and (NUC) in Orlicz function spaces. This answers a question raised by Y. Cui, R. Pluciennik and T. Wang.

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Introduction

Let $(X, \|.\|)$ be a real Banach space, and let $B(X)$ (resp. $S(X)$) be the closed unit ball (resp. the unit sphere) of X. For any subset A of X, we denote by $\operatorname{conv}(A)$, the convex hull of A. Clarkson [2] introduced the concept of uniform convexity. The norm $\|.\|$ is called *uniformly convex* (write (UC)) if for each $\varepsilon > 0$ there is $\delta > 0$ such that for $x, y \in S(X)$ inequality $\|x - y\| > \varepsilon$ implies

$$\left\|\frac{1}{2}(x+y)\right\| < 1 - \delta.$$

For any $x \notin B(X)$, the *drop* determined by x is the set

$$D(x, B(X)) = \operatorname{conv}(\{x\} \cup B(X)).$$

Rolewicz [12] introduced the notion of drop property for Banach spaces. A Banach space X has the *drop property* (write(D)) if for every closed set C disjoint with $B(X)$ there exists an element $x \in C$ such that

$$D(x, B(X)) \cap C = \{x\}.$$

A sequence $\{x_n\} \subset X$ is said to be *ε-separated* for some $\varepsilon > 0$ if

$$\operatorname{sep}(\{x_n\}) = \inf\{\|x_n - x_m\| : n \neq m\} > \varepsilon.$$

A Banach space X is said to be *nearly uniformly convex* (write (NUC)) if for every $\varepsilon > 0$ there exists $\delta \in (0,1)$ such that for every sequence $\{x_n\} \subset B(X)$ with $\operatorname{sep}(\{x_n\}) > \varepsilon$, we have

$$\operatorname{conv}(\{x_n\}) \cap (1-\delta)B(X) \neq \emptyset.$$

A Banach space X is said to have *property* (β) if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\alpha(D(x, B(X))\backslash B(X)) < \varepsilon$$

whenever $1 < \|x\| < 1+\delta$. Here α is the Kuratowski measure of noncompactness on bounded subsets of X. Rolewicz [12] showed that property (β) follows from the uniform convexity and that property (β) implies (NUC). All of these concepts are related as follows:

$$(UC) \Rightarrow (\beta) \Rightarrow (NUC) \Rightarrow (D) \Rightarrow (Rfx), \tag{1}$$

where (Rfx) denotes reflexivity. The implications cannot be reversed in general (see [5], [7], [8], [9], [11], and [12]).

Denote by $\mathbb{R}$ the set of real numbers.

A map $\Phi : \mathbb{R} \to [0, \infty)$ is said to be an *Orlicz function* if Φ is vanishing at 0, even, convex and not identically equal to 0. We say that the Orlicz function Φ satisfies $\triangle_2$*-condition* if there exist a constant $k > 2$ and $u_0 > 0$ such that

$$\Phi(2u) \leq k\Phi(u),$$

for every $|u| \geq u_0$.

Let (G, Σ, μ) be a nonatomic measure space with a finite measure μ. Denote by L^0 the set of all μ-equivalence classes of real valued measurable functions defined on G. Let l^0 stand for the space of all real sequences. By the *Orlicz function space* L_Φ, we mean

$$L_\Phi = \{x \in L^0 : I_\Phi(cx) = \int_G \Phi(cx(t))d\mu < \infty \text{ for some } c > 0\}.$$

Analogously, we define the *Orlicz sequence space* l_Φ by the formula

$$l_\Phi = \{x \in l^0 : I_\Phi(cx) = \sum_{i=1}^{\infty} \Phi(cx_i) < \infty \text{ for some } c > 0\}.$$

L_Φ and l_Φ are equipped with the so called *Luxemburg norm*

$$\|x\| = \inf\{\varepsilon > 0 : I_\Phi(\frac{x}{\varepsilon}) \leq 1\}$$

or with the equivalent norm

$$\|x\|_0 = \inf_{k>0} \frac{1}{k}(1 + I_\Phi(kx))$$

called the *Orlicz norm.* It is well known that for any $x \neq 0$ if, for some k,

$$I_\Psi(p(|kx|)) = 1,$$

where Ψ is the complementary function of Φ and p is the right hand derivative of Φ, then

$$\|x\|_0 = \frac{1}{k}(1 + I_\Phi(kx)).$$

Write L_Φ, l_Φ, L^0_Φ and l^0_Φ for the spaces $(L_\Phi, \|.\|)$, $(l_\Phi, \|.\|)$, $(L_\Phi, \|.\|_0)$, and $(l^0_\Phi, \|.\|_0)$ respectively.

The Orlicz function Φ is *strictly convex* if

$$\Phi(\frac{u+v}{2}) < \frac{\Phi(u) + \Phi(v)}{2}$$

for all $u, v \in \mathbb{R}$, $u \neq v$.
The Orlicz function Φ is said to be *uniformly convex on* $[u_0, \infty)$, where $u_0 > 0$, if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\Phi(\frac{u+v}{2}) \leq (1-\delta)\frac{\Phi(u) + \Phi(v)}{2}$$

holds true for all $u, v \in [u_0, \infty)$ satisfying

$$|u - v| \geq \varepsilon. \max\{u, v\}.$$

For more details we refer to [1] or [9].

In the course of the proof, we use the fact from Theorem 3 and 4 in [3] which states for L_Φ and L^0_Φ that

$$(\beta) \Leftrightarrow \Phi \text{ is uniformly convex on } [u_0, \infty) \text{ for every } u_0 > 0 \text{ and } \Phi \text{ satisfies } \triangle_2\text{-condition.}$$

Results. In [3], it was shown that properties (β), (NUC) and (D) are equivalent for Orlicz sequence space l_Φ, that is the second and the third implication in (1) can be reversed. The authors gave an example showing that the implication (β) $\Rightarrow$ (UC) is not true for spaces l_Φ and l^0_Φ. But they continue to show in contrast to the sequence case that the properties (UC) and (β) are equivalent for Orlicz function spaces L_Φ and L^0_Φ. The only problem left open in the paper concerning the implication in (1) is whether or not (NUC) $\Rightarrow$ (β) in Orlicz spaces L_Φ and L^0_Φ. We show here the answer is affirmative. The proof of the result is mostly based on ingredients in the proofs appearing in [3].

Theorem. *The properties* (β) *and* (*NUC*) *are equivalent for* L_Φ *and* L^0_Φ.

Before we give the proof of the Theorem, we prove a simple but useful result. It is a characterization of uniform convexity of Φ. The author has been informed by the referee that the following lemma is related to some results of S. Chen and H. Hudzik, *On some convexities of Orlicz and Orlicz-Bochner spaces*, Comment. Math. Univ. Carolinae, **29.1** (1988).

Lemma. *For an Orlicz function Φ, Φ is uniformly convex if and only if for any $\varepsilon > 0$ and any $u_0 > 0$, there exists $\delta > 0$ such that for all couples $(u, v) \subset (u_0, \infty)$ satisfying $v - u \geq \varepsilon v$ we have*

$$\Phi(ru + sv) \leq (1 - \delta)(r\Phi(u) + s\Phi(v))$$

for some $r, s \in (0, 1)$ with $r + s = 1$.

Proof: We only prove the "sufficiency". Suppose Φ is not uniformly convex. Thus there exists $\varepsilon > 0$ such that for any $\delta > 0$ there exist u, v with

$$(u, v) \subset (0, \infty), v - u \geq \varepsilon v, \text{ and } p(v) < (1 + \delta)p(u).$$

We now show that

$$\Phi(ru + sv) > (1 - \delta)(r\Phi(u) + s\Phi(v))$$

for all $r, s \in (0, 1)$ with $r + s = 1$.
Write $w = ru + sv$ for such a pair (r, s). Then put

$$\mathrm{I} = \frac{r\Phi(u) + s\Phi(v) - \Phi(u)}{w - u}, \quad \mathrm{II} = \frac{\Phi(w)}{w - u},$$

and

$$\mathrm{III} = \frac{\Phi(w) - \Phi(u)}{w - u}.$$

We estimate

$$\mathrm{I} = \frac{s(\Phi(v) - \Phi(u))}{s(v - u)} = \frac{\Phi(v) - \Phi(u)}{v - u} < (1 + \delta)p(u),$$

$$\mathrm{II} > \frac{\Phi(w) - \Phi(u)}{w - u} > p(u),$$

and

$$\mathrm{III} > p(u).$$

Thus,

$$\frac{r\Phi(u) + s\Phi(v) - \Phi(ru + sv)}{\Phi(ru + sv)} = \frac{\frac{r\Phi(u)+s\Phi(v)-\Phi(u)}{w-u} - \frac{\Phi(ru+sv)-\Phi(u)}{w-u}}{\frac{\Phi(ru+sv)}{w-u}}$$

$$= \frac{\text{I-III}}{\text{II}} < \frac{(1 + \delta)p(u) - p(u)}{p(u)} = \delta.$$

Hence

$$\Phi(ru + sv) > \frac{1}{\delta}(r\Phi(u) + s\Phi(v) - \Phi(ru + sv))$$

and so

$$(1+\frac{1}{\delta})\Phi(ru+sv) > \frac{1}{\delta}(r\Phi(u)+s\Phi(v)).$$

This implies

$$\Phi(ru+sv) > (1-\frac{\delta}{1+\delta})(r\Phi(u)+s\Phi(v)) > (1-\delta)(r\Phi(u)+s\Phi(v)).$$

□

PROOF OF THEOREM: We first consider the space L_Φ. From Theorem 3 in [3] we only need to show that (NUC) implies uniform convexity of Φ on $[u,\infty)$ for all $u>0$. For this, it is enough to show that Φ is strictly convex on $[0,\infty)$ and that there exists $v>0$ such that Φ is uniformly convex on $[v,\infty)$.

If Φ is not strictly convex, we obtain an interval $[a,b]$ in $[0,\infty)$, $G^0\subset G$, $G'\subset G\backslash G^0$ and $c>0$ as in [3] such that Φ is affine on $[a,b]$ and

$$\Phi(\frac{a+b}{2})\mu(G^0)+\Phi(c)\mu(G')=1.$$

Then, for each n, we obtain a partition $\{G_1^n, G_2^n,\dots, G_{2^n}^n\}$ of G^0 such that

$$\mu(G_i^n)=2^{-n}\mu(G^0)\quad (i=1,2,\dots,2^n).$$

Define

$$x_n = a\chi_{E_{1,n}}+b\chi_{E_{2,n}}+c\chi_{G'},$$

where

$$E_{1,n}=\bigcup_{k=1}^{2^{n-1}} G_{2k-1}^n,\ E_{2,n}=\bigcup_{k=1}^{2^{n-1}} G_{2k}^n,\quad (n=1,2,\dots).$$

We show that $\{x_n\}$ violates the property (NUC) by showing that

$$x_n\in B(L_\Phi)\ \text{ for each }\ n\geq 1,\ \mathrm{sep}(\{x_n\}) > \frac{b-a}{\Phi^{-1}(\frac{2}{\mu(G^0)})}$$

and

$$\mathrm{conv}(\{x_n\})\cap(1-\delta)B(L_\Phi)=\emptyset\ \text{ for all }\ \delta>0.$$

Since

$$\begin{aligned} I_\Phi(x_n) &= \frac{\Phi(a)+\Phi(b)}{2}\mu(G^0)+\Phi(c)\mu(G') \\ &= \frac{\Phi(a+b)}{2}\mu(G^0)+\Phi(c)\mu(G')=1, \end{aligned}$$

we first have $\|x_n\| = 1$.
Secondly, we have $\|x_n - y_n\| = \frac{b-a}{\Phi^{-1}(\frac{2}{\mu(G^0)})} > 0$ whenever $n \neq m$.
Finally let $r_1, \ldots, r_n \geq 0$ and $r_1 + \ldots + r_n = 1$. Put $x = r_1x_1 + \ldots + r_nx_n$. Since the values of x on G^0 are convex combinations of a and b with coefficients in $[0,1]$, an easy calculation shows that

$$I_\Phi(x) = \frac{\Phi(a) + \Phi(b)}{2}\mu(G^0) + \Phi(c)\mu(G') = 1.$$

Thus $\|x\| = 1 > 1 - \delta$ for all $\delta > 0$. Therefore Φ is strictly convex.

We now show that Φ is uniformly convex on $[u, \infty)$ for "large" u. Again we suppose for the contrary that there exists $\varepsilon > 0$ such that for any $\delta > 0$ we can find u, v by the Lemma and $G^0 \subset G$ so that $0 < u < v$,

$$\Phi(u)\mu(G) \geq 1, \; v - u \geq \varepsilon v, \; \frac{\Phi(u) + \Phi(v)}{2}\mu(G^0) = 1,$$

and

$$\Phi(ru + sv) > (1 - \delta)(r\Phi(u) + s\Phi(v))$$

for all pairs (r, s) in $(0, 1)$ with $r + s = 1$.
Define

$$x_n = u\chi_{E_{1,n}} + v\chi_{E_{2,n}}$$

where $E_{1,n}$ and $E_{2,n}$ are constructed as above. Again we show that $\{x_n\}$ violates the property (NUC) by showing that

$$x_n \in B(L_\Phi) \;\text{ for each }\; n \geq 1, \; \operatorname{sep}(\{x_n\}) > \frac{\varepsilon}{2}$$

and

$$\operatorname{conv}(\{x_n\}) \cap (1 - \delta)B(L_\Phi) = \emptyset.$$

We estimate for $n, m \geq 1$ and $n \neq m$,

$$I_\Phi(x_n) = (\Phi(u) + \Phi(v))\frac{\mu(G^0)}{2} = 1,$$

and

$$I_\Phi(2\frac{x_n - x_m}{\varepsilon}) = \Phi(2\frac{v - u}{\varepsilon})\frac{\mu(G^0)}{2} \geq \Phi(v)\mu(G^0) > \frac{\Phi(u) + \Phi(v)}{2}\mu(G^0) = 1.$$

Thus $\|x_n\| = 1$ and $\|x_n - x_m\| \geq \frac{\varepsilon}{2}$.
Next let $x = r_1x_1 + \ldots + r_nx_n$ be a convex combination of $x_1, \ldots, x_n$ and estimate

$$I_\Phi(x) > (1 - \delta)(\Phi(u) + \Phi(v))\frac{\mu(G^0)}{2} = 1 - \delta,$$

whence $\|x\| > 1-\delta$.

We consider now the space L^0_Φ. If Φ is not strictly convex, we obtain as in [3], positive numbers a, b, c, and subsets G^0 of G and G' of $G\backslash G^0$ so that p is constant on $[a,b]$, $\mu(G\backslash G^0) > 0$, and

$$\Psi(p(a))\mu(G^0) + \Psi(p(c))\mu(G') = 1.$$

Denote

$$k = 1 + \Phi(\frac{a+b}{2})\mu(G^0) + \Phi(c)\mu(G').$$

Put

$$x_n = \frac{1}{k}(a\chi_{E_{1,n}} + b\chi_{E_{2,n}} + c\chi_{G'}).$$

Since $I_\Psi(p(kx_n)) = 1$, we have

$$\|x_n\|_0 = \frac{1}{k}(1 + I_\Phi(kx_n)) = 1.$$

Also it is seen that for some A with $\mu(A) = \frac{\mu(G^0)}{2}$,

$$\|x_n - x_m\|_0 = \|\frac{a-b}{k}\chi_A\|_0 = \frac{b-a}{k}\|\chi_A\|_0 = \frac{b-a}{k}\mu(A)\Psi^{-1}(\frac{1}{\mu(A)}) > 0$$

whenever $n \neq m$. Now if $x = r_1x_1 + \cdots + r_nx_n$ is a convex combination of $x_1, \ldots, x_n$, we obtain

$$I_\Phi(kx) = \frac{\Phi(a) + \Phi(b)}{2}\mu(G^0) + \Phi(c)\mu(G') = k - 1,$$

and

$$I_\Psi(p(kx)) = \Psi(p(a))\mu(G^0) + \Psi(p(c))\mu(G') = 1.$$

Thus $\|x\|_0 = \frac{1}{k}(1 + I_\Phi(kx)) = 1$. This contradicts the property (NUC).

If Φ is not uniformly convex outside a neighborhood of zero, we can find an $\varepsilon > 0$ such that for each $\delta > 0$ there exist numbers u, v with $v - u \geq \varepsilon v > \varepsilon u > 0$,

$$\frac{\Psi(p(v)) + \Psi(p(u))}{2}\mu(G) \geq 1,$$

and

$$p(v) < (1+\delta)p(u).$$

We choose $G^0 \subset G$ so that

$$\frac{\Psi(p(v)) + \Psi(p(u))}{2}\mu(G^0) = 1$$

and put

$$k = \frac{u(p(u)) + v(p(v))}{2}\mu(G^0).$$

Define

$$x_n = \frac{1}{k}(u\chi_{E_{1,n}} + v\chi_{E_{2,n}}),$$

where $E_{1,n},\ E_{2,n}$ are defined as before.
Since

$$I_\Psi(p(kx_n)) = \frac{\Psi(p(v)) + \Psi(p(u))}{2}\mu(G^0) = 1$$

and

$$I_\Phi(kx_n) = \frac{\Phi(u) + \Phi(v)}{2}\mu(G^0),$$

we see that

$$\|x_n\|_0 = \frac{2 + (\Phi(u) + \Phi(v))\mu(G^0)}{2k} = 1.$$

We also see that for $n \neq m$ we have for some A with $\mu(A) = \frac{\mu(G^0)}{2}$,

$$\begin{aligned}\|x_n - x_m\|_0 = \frac{v-u}{k}\|\chi_A\| &= \frac{v-u}{k}\mu(A)\Psi^{-1}(\frac{1}{\mu(A)}) \\ &= \frac{v-u}{2k}\mu(G^0)\Psi^{-1}(\frac{2}{\mu(G^0)}) > \frac{\varepsilon}{2k}vp(v)\mu(G^0) \geq \frac{\varepsilon}{2}.\end{aligned}$$

This follows from the fact that

$$k = \frac{up(u) + vp(v)}{2}\mu(G^0) \leq vp(v)\mu(G^0),$$

and

$$2 = (\Psi(p(u)) + \Psi(p(v)))\mu(G^0) > \Psi(p(v))\mu(G^0).$$

Now if $x = r_1x_1 + \ldots + r_nx_n$ is a linear convex combination of $x_1, \ldots, x_n$, put

$$y = p(u)\chi_{E_{1,n}} + p(v)\chi_{E_{2,n}}.$$

Thus $I_\Psi(y) = 1$. It is straightforward to see that

$$\begin{aligned}
\|kx\|_0 \geq \int_G kxy &= [p(u)((1+r_n)u + (1-r_n)v) + p(v)((1+r_n)v + (1-r_n)u)]\frac{\mu(G^0)}{4} \\
&\geq [up(u) + vp(v) + (u+v)p(u)]\frac{\mu(G^0)}{4} \\
&= \frac{up(u) + vp(v)}{4}\mu(G^0) + \frac{vp(u) + up(u)}{4}\mu(G^0) \\
&> \frac{k}{2} + (\frac{vp(v)}{1+\delta} + up(u))\frac{\mu(G^0)}{4} \\
&= \frac{k}{2} + \frac{up(u) + vp(v)}{4}\mu(G^0) - \frac{\delta}{1+\delta}vp(v)\frac{\mu(G^0)}{4} \\
&> k - \delta vp(v)\frac{\mu(G^0)}{4} > k - \frac{\delta k}{2}.
\end{aligned}$$

Thus $\|x\|_0 > 1 - \frac{\delta}{2} > 1 - \delta$.
This shows that $\operatorname{conv}(\{x_n\}) \cap (1-\delta)B(L^0_\Phi) = \emptyset$. □

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Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai, 50200 Thailand

E-mail: sompongd@chiangmai.ac.th

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