Bounds for the spectral radius of positive operators

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Abstract. Let f be a non-zero positive vector of a Banach lattice L, and let T be a positive linear operator on L with the spectral radius $r(T)$. We find some groups of assumptions on L, T and f under which the inequalities

$$\sup\{c \geq 0 : Tf \geq c\, f\} \leq r(T) \leq \inf\{c \geq 0 : Tf \leq c\, f\}$$

hold. An application of our results gives simple upper and lower bounds for the spectral radius of a product of positive operators in terms of positive eigenvectors corresponding to the spectral radii of given operators. We thus extend the matrix result obtained by Johnson and Bru which was the motivation for this paper.

Keywords: Banach lattices, positive operators, spectral radius
Classification: 46B42, 47B65, 47A10

1. Introduction and preliminaries

Let L be a (real or complex) Banach lattice of dimension at least two. Let L^+ denote the cone of positive elements in L. The norm of L is said to be a *weakly Fatou norm* if there exists a finite constant $k \geq 1$ such that $0 \leq f_\tau \uparrow f$ in L implies that $\|f\| \leq k \cdot \sup_\tau \|f_\tau\|$, and the norm of L is *order continuous* if $\|f_\alpha\| \downarrow 0$ for any downwards directed system $f_\alpha \downarrow 0$ in L. A Banach lattice L is *Dedekind complete* if every non-empty subset which is bounded from above has a supremum. An element $f \in L^+$ is called a *quasi-interior point* of L^+ if the principal ideal I_f (generated in L by f) is dense in L, and it is said to be a *weak order unit* if the principal band B_f (generated in L by f) is equal to L. The Banach lattice of all bounded linear functionals on L is denoted by L^*. A functional $\varphi \in L^*$ is called *σ-order continuous* if it follows from $f_n \downarrow 0$ in L that $\inf_n |\varphi(f_n)| = 0$, and it is called *order continuous* if $\inf_\alpha |\varphi(f_\alpha)| = 0$ for any downwards directed system $f_\alpha \downarrow 0$ in L. Let L' denote the band of all order continuous functionals in L^*. The Banach lattice L' *separates points* of L whenever for each non-zero $f \in L$ there exists $\varphi \in L'$ such that $\varphi(f) \neq 0$. A functional $\varphi \in L^*$ is said to be *strictly positive* if $\varphi(f) > 0$ for all $f \geq 0$, $f \neq 0$.

By $\mathcal{B}(L)$ we denote the space of all bounded linear operators on a Banach lattice L. An operator $T \in \mathcal{B}(L)$ is called *positive* if $Tf \geq 0$ for all $f \geq 0$. A positive operator $T \in \mathcal{B}(L)$ is called *σ-order continuous* whenever it follows from $f_n \downarrow 0$ in L that $Tf_n \downarrow 0$ in L, and it is called *order continuous* if it follows from $f_\alpha \downarrow 0$ in L that $Tf_\alpha \downarrow 0$ in L. An operator $T \in \mathcal{B}(L)$ is called *power-compact* if T^n is a compact operator for some $n \in \mathbb{N}$. For $T \in \mathcal{B}(L)$, let T^* denote the adjoint operator in $\mathcal{B}(L^*)$. The spectral radius of an operator T is denoted by $r(T)$. Here, in case of a real Banach lattice, we understand the spectral radius of the canonical extension of T to the complexification of L. A positive operator $T \in \mathcal{B}(L)$ is called *band irreducible*, if T leaves no band in L invariant except $\{0\}$ and L itself, and it is said to be *irreducible*, if T leaves no closed ideal in L invariant except $\{0\}$ and L itself. For notions not explained in the text, we refer the reader to the books of Zaanen [12], Schaefer [9], and Aliprantis, Burkinshaw [1].

Throughout the paper L denotes a Banach lattice. Let $f, g \in L^+$, and assume that g is non-zero. Define the following sets

$$\Delta(f,g) = \{c \geq 0 \ : \ f \geq c\, g\} \ \text{ and } \ \Sigma(f,g) = \{c \geq 0 \ : \ f \leq c\, g\}.$$

Since L is Archimedean, one can show easily that $\Delta(f,g)$ is a non-empty bounded closed interval, and so we can define $\delta(f,g) = \max \Delta(f,g)$. Also, we set $\sigma(f,g) = \min \Sigma(f,g)$ if $\Sigma(f,g)$ is non-empty, and $\sigma(f,g) = \infty$ otherwise.

Let T be a positive operator on L, and let $f \in L^+$ be a non-zero element. The numbers $\delta(Tf,f)$ and $\sigma(Tf,f)$ are known as the lower and upper *Collatz-Wielandt numbers*, respectively. In the present paper we study the following inequalities

$$\delta(Tf,f) \leq r(T) \leq \sigma(Tf,f).$$

These inequalities were considered by some authors, even in more general setting of ordered Banach space (see [7], [2] and [8]). While the left-hand inequality always holds, the right-hand inequality is not true even under the assumption that f is a quasi-interior point of L^+. We obtain some results under various assumptions on L, T and f. An application of our results gives simple upper and lower bounds for the spectral radius of a product of positive operators. We thus extend the matrix result obtained by Johnson and Bru [5] which was our motivation for this work.

Let T be a order continuous operator on L. It is easily seen that L' is invariant under the adjoint T^*. The restriction $T^*|_{L'}$ of T^* to L' is denoted by T'.

Proposition 1.1. *Assume that L is a Dedekind complete lattice with a weakly Fatou norm, and that L' separates points of L. If T is an order continuous operator on L, then $r(T') = r(T)$.*

Proof: It follows from Theorem 107.7 of [12] (see also the equality (2) on p. 393 of [12]) that L can be (not necessarily isometrically) embedded into $(L')'$ as a Banach space. Then we have

$$r(T) \geq r(T') \geq r((T')') \geq r(T),$$

so that $r(T') = r(T)$. □

Proposition 1.2. *Let f be a weak unit of L^+. Let I be the closed ideal of L generated by f. Then the positive operator $\gamma : L' \to I'$ defined by $\gamma(\varphi) = \varphi|_I$ is an isometric isomorphism of Banach lattices.*

Proof: Assume that $\varphi|_I = 0$ for some $\varphi \in L'$ and choose $g \in L^+$. Put $g_n = \inf\{g, nf\}$ for each $n \in \mathbb{N}$. Then $g_n \in I$ and $(g - g_n) \downarrow 0$. Since $\varphi \in L'$, we have $\inf_n |\varphi(g - g_n)| = 0$, and so $\varphi(g) = 0$. Now the equality $L = L^+ - L^+$ implies that $\varphi = 0$. Hence γ is injective.

To prove surjectivity of γ, pick $\psi \in I'$. By [12, Theorem 83.7] there exist positive functionals $\varphi_1, \varphi_2 \in L'$ such that $\varphi_1|_I = \psi^+$ and $\varphi_2|_I = \psi^-$. Then $\varphi := \varphi_1 - \varphi_2 \in L'$ and $\varphi|_I = \psi$. The last consideration also shows that the operator γ^{-1} is positive. Indeed, if ψ is positive, then its extension φ is also positive. It is well-known that this implies that γ is a Riesz isomorphism (see e.g. [1, Theorem 7.3]).

Let us show that γ is isometric. Pick $\varphi \in L'$. It is clear that $\|\varphi|_I\| \le \|\varphi\|$. So, we have to show that $\|\varphi|_I\| \ge \|\varphi\|$. Choose $\varepsilon > 0$ and $g \in L$ with norm 1 such that $|\varphi(g)| \ge \|\varphi\| - \varepsilon$. Define $u_n = \inf\{g^+, nf\}$, $v_n = \inf\{g^-, nf\}$, and $g_n = u_n - v_n$, Then $g_n \in I$ for each $n \in \mathbb{N}$, and the sequence $\{g_n\}_{n\in\mathbb{N}}$ order converges to g, since $u_n \uparrow g^+$ in $v_n \uparrow g^-$. From $|g_n| \le u_n + v_n \le g^+ + g^- = |g|$ it follows that $\|g_n\| \le \|g\| = 1$. We thus have $|\varphi(g_n)| \le \|\varphi|_I\| \|g_n\| \le \|\varphi|_I\|$. On the other hand, since $\inf_n |\varphi(g - g_n)| = 0$, there is $n \in \mathbb{N}$ such that $|\varphi(g_n)| \ge |\varphi(g)| - \varepsilon$. We therefore conclude that $\|\varphi|_I\| \ge |\varphi(g)| - \varepsilon \ge \|\varphi\| - 2\,\varepsilon$. This shows that $\|\varphi|_I\| \ge \|\varphi\|$. ☐

The following result was actually shown in the proof of Theorem 5 of [4].

Theorem 1.3. *Let $T > 0$ be a σ-order continuous, power-compact operator on L with $r(T) > 0$. Then there exists a σ-order continuous, non-zero positive functional $\varphi \in L^*$ satisfying $T^*\varphi = r(T)\,\varphi$. If, in addition, T is band irreducible, φ can be chosen to be strictly positive.*

In [4] the following extension of the famous Jentzsch-Perron theorem is also proved.

Theorem 1.4. *Let T be a positive σ-order continuous, band irreducible, power-compact operator on L. Then $r(T) > 0$ and $r(T)$ is an eigenvalue of T of algebraic multiplicity one. Furthermore, the eigenspace contains a weak order unit.*

2. On Collatz-Wielandt bounds

The first part of the next result is known even in more general setting of ordered Banach spaces (see [6], [7], [2], and [8]). We include here its proof because of its simplicity. The second part is concerned with the case of equality, and it seems to be new.

Theorem 2.1. *Let T be a positive operator on a Banach lattice L, and let $f \in L^+$ be a non-zero element. Then*

$$\delta(Tf, f) \le r(T).$$

If T is a σ-order continuous, band irreducible, power-compact operator, then there is an equality in this inequality if and only if f is a positive eigenfunction of T pertaining to $r(T)$.

Proof: By a successive application of $T \geq 0$ to the inequality $Tf \geq \delta f$, where $\delta = \delta(Tf, f)$, we obtain $T^n f \geq \delta^n f$ for all $n \in \mathbb{N}$, so that $\delta^n \|f\| \leq \|T^n f\| \leq \|T^n\| \|f\|$. This gives $\delta \leq \|T^n\|^{1/n}$ for all $n \in \mathbb{N}$, and hence $\delta \leq r(T)$.

To prove the second assertion, assume that $\delta(Tf, f) = r(T)$, that is $Tf - r(T)f \geq 0$. By Theorem 1.4, $r(T) > 0$, and so Theorem 1.3 gives a strictly positive functional $\varphi \in L^*$ satisfying $T^* \varphi = r(T)\varphi$. We then have

$$\varphi(Tf - r(T)f) = (T^* \varphi)f - r(T)\,\varphi(f) = 0.$$

It follows that $Tf = r(T)f$, and the theorem is proved. □

Motivated by Proposition 2 of [3] we introduce the following class of operators. We say that a positive operator T on L has *property* (p) if $r(T)$ belongs to the closure of the set $(-\infty, r(T)) \cap \rho(T)$, where $\rho(T)$ denotes the resolvent set of T.

Theorem 2.2. *Suppose that L is a Dedekind complete lattice with a weakly Fatou norm, and that L' separates points of L. Let T be an order continuous positive operator on L having property (p), and let $f \in L^+$ be a weak order unit. Then*

$$r(T) \leq \sigma(Tf, f).$$

Proof: We assume on the contrary that $\sigma := \sigma(Tf, f) < r(T)$. The principal ideal $I_f = \{g \in L : |g| \leq \lambda f \text{ for some } \lambda \geq 0\}$ is an AM-space under the norm defined by $\|g\|_0 := \sigma(|g|, f)$. If $g \in I_f \cap L^+$, then $Tg \leq \|g\|_0 Tf \leq \sigma \|g\|_0 f$ implies that I_f is invariant under T and $\|Tg\|_0 \leq \sigma \|g\|_0$. Denoting by T_0 the restriction of T on I_f, we have $\|T_0\|_0 \leq \sigma$. Therefore, for any $\lambda \in (\sigma, r(T))$, the resolvent $(\lambda - T_0)^{-1}$ exists and it is a positive operator on I_f. By the assumption there exists $\lambda \in (\sigma, r(T))$ that belongs to the resolvent set of T. Let I be the closure of I_f in L. Then the resolvent $(\lambda - T|_I)^{-1}$ is a positive operator on I. This implies that $\lambda \geq r(T|_I)$ (see e.g. [9, Exercise V.5] or [10, Lemma 2]). We claim that $r(T|_I) = r(T)$ which then gives a contradiction. By Proposition 1.2 the map $\varphi \to \varphi|_I$ is an isometric isomorphism of Banach lattices L' and I'. It follows that

$$r(T') = r((T|_I)') \leq r(T|_I) \leq r(T).$$

Since $r(T') = r(T)$ by Proposition 1.1, we conclude that $r(T|_I) = r(T)$ as desired. □

The following example shows that in Theorem 2.2 the assumption that T is an order continuous operator cannot be omitted.

Example 2.3. Let ϕ be a Banach limit on l^∞, and $e = (1, 1, 1, \dots) \in l^\infty$. By $Tf = \phi(f)\, e$ we define a positive compact operator on l^∞ with $r(T) = \phi(e) = 1$.

Letting $f = (1, 1/2, 1/3, 1/4, \dots) \in l^\infty$, we have $Tf = 0$, and so $0 = \sigma(Tf, f) < r(T) = 1$.

The following result on the upper bound for the spectral radius follows also from [3, Proposition 2] and [2, Proposition 2.3].

Theorem 2.4. *Let T be a positive operator on L having property (p), and let $f \in L^+$ be a quasi-interior point. Then*

$$r(T) \le \sigma(Tf, f).$$

Proof: The proof goes along the lines of the last proof, except that the equality $r(T|_I) = r(T)$ is trivially true, since $I = L$ by the assumption on f. □

Theorem 2.4 is not true without the assumption that T has property (p).

Example 2.5. The backward shift S on l^2 defined by $S(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots)$ has the spectral radius $r(S) = 1$. However, $f = (1, 1/2, 1/4, 1/8, \dots) \in l^2$ is a quasi-interior point satisfying $\sigma(Sf, f) = 1/2$.

Under a strong assumption on T we obtain the following

Theorem 2.6. *Let T be a positive σ-order continuous, power-compact operator on a Banach lattice L, and let $f \in L^+$ be a weak order unit. Then*

$$r(T) \le \sigma(Tf, f).$$

If, in addition, T is a band irreducible operator, then there is an equality if and only if f is an eigenfunction of T corresponding to $r(T)$.

Proof: We clearly may assume that $r(T) > 0$. Suppose on the contrary that $\sigma(Tf, f) = r(T) - c$ for some $c > 0$. By Theorem 1.3 there exists a non-zero σ-order continuous functional $\varphi \in L^*$ such that $T^*\varphi = r(T)\varphi$. Then

$$\varphi(r(T)f - Tf) = r(T)\,\varphi(f) - (T^*\varphi)f = 0. \tag{1}$$

Since $r(T)f - Tf \ge cf$, we obtain $\varphi(f) = 0$ which easily yields $\varphi(g) = 0$ for all $g \in B_f$ as φ is σ-order continuous. Since $B_f = L$, we get $\varphi = 0$ which is a contradiction.

Suppose now that T is a band irreducible operator. Then $r(T) > 0$ by Theorem 1.4. If there is an equality in the inequality, it follows from (1) that $r(T)f - Tf = 0$, i.e. f is an eigenelement of T pertaining to $r(T)$, because φ can be chosen to be strictly positive by Theorem 1.3. □

An application of Theorem 1.4 gives the following consequence of Theorems 2.1 and 2.6 that is known under some similar groups of assumptions, even in the ordered Banach space setting (see [6], [7], [2], and [8]).

Theorem 2.7. *Let L be a Banach lattice with non-empty set W of all weak order units. Let T be a σ-order continuous, band irreducible, power-compact operator on L. Then*

$$r(T) = \max\{\delta(Tf, f) : f \in W\} = \min\{\sigma(Tf, f) : f \in W\}.$$

Moreover, either extremum is attained for some $f \in W$ if and only if $f \in W$ is the (essentially unique) eigenelement of T.

Theorem 2.7 can be viewed as a generalization of the minimax theorem for the spectral radius of an irreducible non-negative matrix (due to Wielandt [11]). The special case of Theorem 2.7 is following

Corollary 2.8. *Let (X, μ) be a σ-finite measure space, and let T be a power-compact operator on $L^p(X, \mu)$ $(1 \le p \le \infty)$. If $p < \infty$, assume T to be irreducible; if $p = \infty$, assume T to be the adjoint of an irreducible operator on $L^1(X, \mu)$. If W denotes the set of all, almost everywhere positive functions in $L^p(X, \mu)$, then*

$$r(T) = \max\{\delta(Tf, f) : f \in W\} = \min\{\sigma(Tf, f) : f \in W\}.$$

Moreover, either extremum is attained for some $f \in W$ if and only if $f \in W$ is the (essentially unique) eigenfunction of T.

PROOF: Since every closed ideal of $L^p(X, \mu)$ $(1 \le p < \infty)$ is a band, every positive operator on $L^p(X, \mu)$ is σ-order continuous (see [1, Theorem 4.8]). The same argument shows that T is irreducible. The assumptions of Theorem 2.7 are satisfied in the case $p = \infty$ as well. Indeed, it can be shown that the adjoint of a σ-order continuous, (band) irreducible operator on $L^1(X, \mu)$ is also σ-order continuous and band irreducible. □

Essentially, Corollary 2.8 was proved in Schaefer [10, Theorem 1], because it is not difficult to show that the numbers introduced there coincide with the Collatz-Wielandt numbers. (The proof of this surprising fact is left to the reader.) The papers [10] and [4] influenced us that we considered the Banach lattice setting. As mentioned before, the same upper bounds for the spectral radius of positive operators are also studied in the ordered Banach space setting, where similar assumptions on L, T, and f are considered. However, it seems that Theorems 2.2, 2.6 and 2.7 are not special cases of those results.

3. Bounds for the spectral radius of a product of positive operators

In general, there is no relation between the spectral radius of a product of positive operators and the product of the respective spectral radii. In this section Theorems 2.1, 2.2, 2.4 and 2.6 are applied in order to obtain simple upper and lower bounds for the spectral radius of a product of positive operators in terms of positive eigenvectors corresponding to the spectral radii of given operators. We thus generalize the matrix result due to Johnson and Bru [5].

Theorem 3.1. *Let $T_1, T_2, \dots, T_n$ be positive operators on a Banach lattice L with strictly positive spectral radii. Suppose that there exist non-zero positive elements $f_1, f_2, \dots, f_n$ such that $T_i\, f_i = r(T_i)\, f_i$ for $i = 1, 2, \dots, n$. Then*

$$\delta(f_n, f_{n-1})\, \delta(f_{n-1}, f_{n-2}) \dots \delta(f_2, f_1)\, \delta(f_1, f_n) \le \frac{r(T_1 T_2 \dots T_n)}{r(T_1)\, r(T_2) \dots r(T_n)}\ .$$

Proof: Since $f \ge \delta(f, g)\, g$ for any non-zero positive elements $f, g \in L$, we have

$$\begin{aligned}(T_1 \dots T_n) f_n &= r(T_n)\, (T_1 \dots T_{n-1}) f_n \ge r(T_n)\, \delta(f_n, f_{n-1})\, (T_1 \dots T_{n-1}) f_{n-1} \\ &\ge r(T_n)\, r(T_{n-1})\, \delta(f_n, f_{n-1})\, \delta(f_{n-1}, f_{n-2})\, (T_1 \dots T_{n-2}) f_{n-2} \ge \cdots \\ &\dots \ge r(T_n)\, r(T_{n-1}) \dots r(T_2)\, \delta(f_n, f_{n-1})\, \delta(f_{n-1}, f_{n-2}) \dots \delta(f_2, f_1)\, T_1 f_1 \\ &\ge r(T_1)\, r(T_2) \dots r(T_n)\, \delta(f_n, f_{n-1})\, \delta(f_{n-1}, f_{n-2}) \dots \delta(f_2, f_1)\, \delta(f_1, f_n)\, f_n.\end{aligned}$$

It follows that

$$\delta((T_1 \dots T_n) f_n, f_n) \ge r(T_1) \dots r(T_n)\, \delta(f_n, f_{n-1}) \dots \delta(f_2, f_1)\, \delta(f_1, f_n).$$

Theorem 2.1 now completes the proof. □

It should be noted that Theorem 3.1 (as Theorem 2.1) can be generalized to the ordered Banach space setting. However, the situation with the upper bound for the spectral radius is much more involved.

Theorem 3.2. *Let $T_1, T_2, \dots, T_n$ be positive operators on a Banach lattice L with strictly positive spectral radii, and let $f_1, f_2, \dots, f_n$ be weak order units of L such that $T_i\, f_i = r(T_i)\, f_i$ for $i = 1, 2, \dots, n$. Let the operator $T = T_1 T_2 \dots T_n$ have property (p). Assume also that either*

(i) *L is a Dedekind complete lattice with a weakly Fatou norm, L' separates points of L, and T is σ-order continuous, or*
(ii) *at least one of the elements $f_1, \dots, f_n$ is a quasi-interior point of L^+, or*
(iii) *T is σ-order continuous power-compact operator.*

Then

$$\frac{r(T_1 T_2 \dots T_n)}{r(T_1)\, r(T_2) \dots r(T_n)} \le \sigma(f_n, f_{n-1})\, \sigma(f_{n-1}, f_{n-2}) \dots \sigma(f_2, f_1)\, \sigma(f_1, f_n).$$

Proof: We clearly may assume that $\sigma(f_{i+1}, f_i) < \infty$ for all $i = 1, 2, \dots, n$ (letting $f_{n+1} = f_1$). It follows that in case (ii) all of the elements $\{f_k\}_{k=1}^n$ are quasi-interior points of L^+. Then the calculation is similar as in the proof of Theorem 3.1 except for that we use the inequality $f \le \sigma(f, g)\, g$ (where f and g are weak order units with $\sigma(f, g) < \infty$), the direction of the inequalities is reversed, and the upper bound is a consequence of Theorems 2.2, 2.4 and 2.6.

□

The following examples show that in Theorem 3.2 we cannot drop neither the conditions (i), (ii), and (iii) nor the assumption that T has property (p). It turned out that seeking for counterexamples was not an easy job.

Example 3.3. Let e, f and T be as in Example 2.3. Let L be the Banach lattice $l^\infty \times l^\infty$ with the norm $\|(u,v)\| = \max\{\|u\|_\infty, \|v\|_\infty\}$. Define the operators T_1 and T_2 on L by

$$T_1 = \begin{bmatrix} I & T \\ 0 & I \end{bmatrix} \quad \text{and} \quad T_2 = \begin{bmatrix} I & 0 \\ T & I \end{bmatrix},$$

where I denotes the identity operator on l^∞. Then (f,f) is a weak order unit of L which is an eigenvector of both T_1 and T_2 corresponding to the eigenvalue $r(T_1) = r(T_2) = 1$. It is easy to see that $((1+\sqrt{5})e, 2e)$ is an eigenvector of T_1T_2 belonging to the eigenvalue $(3+\sqrt{5})/2$, so that $r(T_1T_2) \geq (3+\sqrt{5})/2 > 1$. Since T_1T_2 is a compact perturbation of the identity on L, it has property (p).

Example 3.4. Let f and S be as in Example 2.5, and let operators T_1 and T_2 on $L := l^2 \times l^2$ be defined by

$$T_1 = \begin{bmatrix} S & S \\ 0 & I \end{bmatrix} \quad \text{and} \quad T_2 = \begin{bmatrix} I & 0 \\ S & S \end{bmatrix},$$

where I is the identity on l^2. Then (f,f) is a quasi-interior point of L^+ that is an eigenvector of both T_1 and T_2 corresponding to the eigenvalue $r(T_1) = r(T_2) = 1$. Letting $g = (1, 0.9, 0.9^2, 0.9^3, \dots) \in l^2$ a short computation shows that $(3g, 2g)$ is an eigenvector of T_1T_2 pertaining to the eigenvalue $9/4$. Hence $r(T_1T_2) \geq 9/4 > 1$.

Acknowledgments. The author would like to express his gratitude to Professor Matjaž Omladič for his helpful advice. He also thanks the Research Ministry of Slovenia for the support.

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(*Received* September 14, 1998, *revised* November 8, 1999)